Exercise on properties of holomorphic functions

The goal of this exercise is to solidify your understanding of some theorems about the local and global properties of holomorphic functions: (1) the open mapping theorem, (2) the maximum modulus theorem, (3) Rouché's theorem, and (4) the Schwarz lemma.

1. The set $\{\, z \in \mathbb{C} : |\operatorname{Re} z| < |\operatorname{Im} z| \,\}$ is an open set, and the function f defined by $f(z) = (\operatorname{Im} z)/|\operatorname{Im} z|$ is holomorphic on the set and not constant; yet the range of f is not open. Why doesn't this contradict the open mapping theorem?

2. Consider the exponential function on the right half-plane $\{\, z \in \mathbb{C} : \operatorname{Re} z > 0 \,\}$. The maximum of $|\exp(z)|$ for z in the boundary of the right half-plane is 1, yet $|\exp(z)|$ is unbounded on the right-half plane. Why doesn't this contradict the maximum modulus theorem?

3. Let f be the constant function 2, and let g be the constant function -1. Then f and g have the same number of zeroes inside the unit disc (namely, none), yet the inequality

$$|f(z) - g(z)| < |f(z)| + |g(z)|$$

is false at all boundary points of the unit disc. Why doesn't this contradict Rouché's theorem?

4. Suppose that f is a holomorphic function from the unit disc into itself. If $f(0) = 1/2$, how big can $|f'(0)|$ be? Can you find an example that achieves your upper bound?