Instructions Do any five of the following six problems. You may consult the textbook but not other sources; in particular, you may not ask another person for help solving the problems. You may cite and use results proved in the textbook. Please submit your solutions to me in my office before 4:00 P.M. on Friday, September 29.

1. Let E denote the open subset of the complex plane defined by $E := \{ z \in \mathbb{C} : |\sin(z)| < |z| \}$. Show that the area of the set E is infinite.

2. Solve exercise 2.4 in the textbook: namely, derive the Cauchy-Riemann equations in polar coordinates.

3. We know that a power series converges absolutely in a certain disk. Consider instead a *Dirichlet series* of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z}, \tag{1}$$

where the complex numbers a_n are constants (independent of the variable z), and the expression n^z means, by definition, $\exp(z \ln n)$, where ln denotes the natural logarithm of a positive real number. Set

$$A := \limsup_{n\to\infty} \frac{\ln |a_n|}{\ln n}.$$

Supposing that A is finite, show that the Dirichlet series (1) converges absolutely when $\operatorname{Re} z > A + 1$.

4. The power series $1 - z + z^2 - z^3 + \cdots$ is a geometric series that converges to $1/(1 + z)$ when $|z| < 1$. Consequently, one expects that the formal anti-derivative

$$L(z) := z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

should have the properties of a logarithm of $1 + z$. Prove that indeed $\exp(L(z)) = 1 + z$ when $|z| < 1$.

You may assume that a power series can be differentiated term by term inside the open disk of convergence (a fact that we have stated but not yet officially proved).

5. Let C be a continuously differentiable simple closed curve equipped with the standard counterclockwise orientation. Show that $\int_C (\operatorname{Im} z)\,dz$ equals the negative of the area of the region enclosed by the curve C.

6. Suppose f is an analytic function in the unit disk $\{\, z \in \mathbb{C} : |z| < 1 \,\}$. Then, by definition, the function f has a derivative. This problem asks you to show that the function f also *is* a derivative. Namely, set

$$F(z) := \int_0^1 z f(tz)\,dt \qquad \text{when } |z| < 1.$$

Prove that F is differentiable and that $F'(z) = f(z)$.