The theorem of Mertens about the Cauchy product of infinite series

If the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge absolutely, then one can freely rearrange terms to find that

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n, \qquad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k}. \tag{1}$$

Franz Carl Joseph Mertens (1840–1927) observed[1] that (1) still holds when only one of the first two series, say $\sum_n a_n$, converges absolutely, as long as the second series $\sum_n b_n$ converges conditionally. The argument of Mertens goes as follows.

Proof. Let A_n, B_n, and C_n denote the partial sums $\sum_{k=0}^{n} a_k$, $\sum_{k=0}^{n} b_k$, and $\sum_{k=0}^{n} c_k$. It suffices to prove that both (i) $\lim_{n\to\infty}(C_{2n} - A_n B_n) = 0$ and (ii) $\lim_{n\to\infty}(C_{2n+1} - A_{n+1} B_n) = 0$. For (i), observe that $C_{2n} - A_n B_n$ equals

$$\begin{aligned} &a_0(b_{n+1} + b_{n+2} + \cdots + b_{2n}) + a_1(b_{n+1} + b_{n+2} + \cdots + b_{2n-1}) + \cdots + a_{n-1}b_{n+1} \\ &+ a_{n+1}(b_0 + b_1 + \cdots + b_{n-1}) + a_{n+2}(b_0 + b_1 + \cdots + b_{n-2}) + \cdots + a_{2n}b_0. \end{aligned} \tag{2}$$

By hypothesis, there are numbers A and B such that $\sum_{j=0}^{m} |a_j| < A$ and $\left|\sum_{j=0}^{m} b_j\right| < B$ for all m. Fix a positive ϵ. By hypothesis, there exists a number N such that when $n \geq N$ and $m \geq 1$, one has

$$\sum_{j=n+1}^{n+m} |a_j| < \frac{\epsilon}{A+B} \qquad \text{and} \qquad \left|\sum_{j=n+1}^{n+m} b_j\right| < \frac{\epsilon}{A+B}.$$

Now (2) shows that when $n \geq N$, one has that $|C_{2n} - A_n B_n| < A \cdot \frac{\epsilon}{A+B} + B \cdot \frac{\epsilon}{A+B} = \epsilon$. Thus $\lim_{n\to\infty}(C_{2n} - A_n B_n) = 0$ as claimed.

To establish the limit (ii), observe that $C_{2n+1} - A_{n+1} B_n$ equals

$$\begin{aligned} &a_0(b_{n+1} + b_{n+2} + \cdots + b_{2n+1}) + a_1(b_{n+1} + b_{n+2} + \cdots + b_{2n}) + \cdots + a_n b_{n+1} \\ &+ a_{n+2}(b_0 + b_1 + \cdots + b_{n-1}) + a_{n+3}(b_0 + b_1 + \cdots + b_{n-2}) + \cdots + a_{2n+1}b_0, \end{aligned}$$

and argue analogously to case (i). □

[1] F. Mertens, Ueber die Multiplicationsregel für zwei unendliche Reihen, *Journal für die Reine und Angewandte Mathematik* **79** (1874) 182–184.