The second problem on the third homework assignment asked you to show that there are infinitely many complex numbers z such that $\exp(z) = z$. Here are some remarks about this problem.

1. This problem is the hardest one that I have assigned so far.
2. The problem has more than one interpretation. You could view the problem as being a statement about fixed points of the exponential function, or you could view the problem as a statement about zeroes of the function $\exp(z) - z$.
3. There is a natural plan for solving the problem—translate it into a pair of simultaneous nonlinear equations in the underlying real variables—but executing the plan is not trivial.
4. Similar problems have appeared in the past on the PhD qualifying examination in complex analysis, with the idea that the problem can be solved by applying sophisticated techniques from Math 618.

Here is a solution from first principles, using only tools that have been covered so far in the course or that you know from prerequisite courses. I shall show that there are infinitely many complex numbers z in the first quadrant such that $\exp(z) = z$.

Indeed, I claim there are sequences $\{x_n\}$ and $\{y_n\}$ of positive real numbers such that

$$e^{2x_n} = x_n^2 + y_n^2 \qquad \text{and} \qquad y_n = \arctan(y_n/x_n) + 2\pi n. \tag{1}$$

Here I am using the notation $\arctan(t)$ to denote the unique number between $-\pi/2$ and $\pi/2$ whose tangent equals the real number t.

[Remark: This is the first time in the course that the notion of "branches" arises. The tangent function is periodic, hence not one-to-one, so it is necessary to restrict the domain of the tangent function in order to define an inverse function. There are infinitely many choices of how to restrict the domain; the domain $(-\pi/2, \pi/2)$ gives rise to the "principal branch" of the arctangent function. Since the exponential function is periodic (with period $2\pi i$), there is similarly no unique way to define an inverse of $\exp(z)$. We shall spend some time later in the course coming to grips with the problem of giving a meaning to the symbol $\log(z)$ when z is a complex number.]

Suppose for the moment that the existence of sequences $\{x_n\}$ and $\{y_n\}$ satisfying (1) has been verified. Set z_n equal to $x_n + iy_n$, and for simplicity, let θ_n denote $\arctan(y_n/x_n)$. Then the first equation in (1) implies that

$$\exp(z_n) = e^{x_n}(\cos(y_n) + i\sin(y_n)) = |z_n|(\cos(y_n) + i\sin(y_n)),$$

and the second equation in (1) implies that

$$\exp(z_n) = |z_n|(\cos(\theta_n + 2\pi n) + i\sin(\theta_n + 2\pi n)) = |z_n|(\cos(\theta_n) + i\sin(\theta_n)) = z_n.$$

Notice that the complex numbers z_n for different values of n are distinct, for $|y_j - y_k| > \pi$ when $j \neq k$. Thus verifying the existence of the sequences $\{x_n\}$ and $\{y_n\}$ will solve the problem.

To make this verification, I first make the subsidiary claim that for every positive integer n, there exists a positive real number x_n such that

$$\sqrt{e^{2x_n} - x_n^2} - \arctan(x_n^{-1}\sqrt{e^{2x_n} - x_n^2}\,) = 2\pi n. \tag{2}$$

Notice that $e^x > x$ when x is a real number, hence $e^{2x} - x^2 > 0$, so the expression $\sqrt{e^{2x_n} - x_n^2}$ represents a well-defined positive real number, call it y_n. For this choice of y_n, it is evident that (2) implies (1).

Finally, to verify the existence of a number x_n satisfying (2), observe that the function

$$\sqrt{e^{2x} - x^2} - \arctan(x^{-1}\sqrt{e^{2x} - x^2}\,) \tag{3}$$

is a continuous function of x when $x > 0$ that has a one-sided limit as $x \to 0^+$ equal to the negative value $1 - \frac{1}{2}\pi$. Moreover, this function tends to ∞ when $x \to \infty$ (because the expression $e^{2x} - x^2$ blows up, while the arctangent function stays bounded above by $\pi/2$). The intermediate-value theorem from real calculus implies that the function (3) has every positive real number in its range. Consequently, there is a positive real number x_n at which the function (3) takes the value $2\pi n$. This conclusion completes the verification of the subsidiary claim and also completes the solution of the problem.

Remarks:

1. Notice that the preceding discussion can be turned into a method for computing solutions numerically. All that is needed is to approximate the values of the function (3).

2. A more sophisticated solution of the problem, using tools from Math 618 that you do not know yet, could go as follows. Suppose if possible that the function $\exp(z) - z$ has only a finite number of zeroes (conceivably none at all). These zeroes can be factored out to write $\exp(z) - z$ as the product of a polynomial $p(z)$ and a zero-free entire function. Now a zero-free entire function has an entire logarithm. In other words, there is an entire function $f(z)$ such that

   $$\exp(z) - z = p(z)\exp(f(z)). \tag{4}$$

   Modifying the polynomial p by a constant factor shows that there is no loss of generality in supposing that $f(0) = 0$. Hadamard's factorization theorem for entire functions of finite order implies that $f(z)$ must have the form az for some complex number a. Considering growth rates of the two sides of (4) as $z \to \pm\infty$ and as $z \to \pm i\infty$ shows that a must equal 1. Dividing both sides of (4) by $\exp(z)$ then shows that the function $z\exp(-z)$ is equal to a polynomial. Evidently that polynomial must take the value 0 at 0, and dividing out the factor of z shows that $\exp(-z)$ itself is a polynomial, an absurd conclusion. Thus the assumption that the function $\exp(z) - z$ has only a finite number of zeroes leads to a contradiction.