Examination 2

1. Suppose f is a holomorphic function in the right-hand half-plane such that $f'(z) = 1/z$ when $\mathrm{Re}(z) > 0$, and $f(1) = i$. Find the value of $f(1+i)$ in the form $a + bi$.

2. Suppose f has an isolated singularity at 0, and the residue of f at 0 is equal to 4. Suppose $g(z) = f(2z) + 3f(z)$ for all z in a punctured neighborhood of 0. Find the residue of g at 0.

3. Suppose $\gamma_0 \colon [0,1] \to \mathbb{C} \setminus \{0\}$ is a differentiable closed curve lying in the punctured plane, and $\gamma_1(t) = \gamma_0(t^2)$ when $0 \le t \le 1$. If the index (winding number) of γ_0 about the origin is equal to 5, what is the value of the index of γ_1 about the origin? Explain how you know.

4. Find the maximum value of $|i + z^2|$ when $|z| \le 2$.

5. Prove that $\displaystyle\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{\pi}{2\sqrt{2}}$.
   (This integral can—in principle—be evaluated by using techniques of real calculus, but you are more likely to be successful by applying the residue theorem.)

6. Let V denote an open subset of $\mathbb{C}$, let γ denote a differentiable simple closed curve that lies in V, and let f denote a holomorphic function in V. Answer any two of the following three questions.
   (a) What additional property of γ is necessary and sufficient to guarantee that $\int_\gamma f(z)\,dz = 0$ for every f?
   (b) What additional property of f is necessary and sufficient to guarantee that $\int_\gamma f(z)\,dz = 0$ for every γ?
   (c) What additional property of V is necessary and sufficient to guarantee that $\int_\gamma f(z)\,dz = 0$ for every f and every γ?