Preview of the Final Examination

The final examination takes place in the usual classroom from 3:00 to 5:00 on the afternoon of Friday, December 12. Please bring your own paper to the exam.

Part I I will choose six of the following twelve items and ask you to state three of those six.

- The Cauchy–Riemann equations.
- The formula for the radius of convergence of a power series.
- Some version of Cauchy's integral formula that applies to disks.
- Some version of Cauchy's theorem that applies to an annulus.
- Morera's theorem.
- Cauchy's estimate for derivatives at the center of a disk.
- Liouville's theorem.
- Some version of the maximum modulus theorem.
- The argument principle.
- Some version of Rouché's theorem.
- Some version of the open mapping theorem.
- The residue theorem.

Part II I will choose six of the following twelve problems and ask you to solve three of those six.

- Problem 1 on the August 2008 qualifying exam: Find the Laurent series of $\dfrac{1}{z(z-1)(z-2)}$ valid in the annulus $\{\, z \in \mathbb{C} : 1 < |z| < 2 \,\}$.
- Problem 3 on the August 2008 qualifying exam: If f and g are zero-free holomorphic functions on the unit disk such that $\dfrac{f'(1/n)}{f(1/n)} = \dfrac{g'(1/n)}{g(1/n)}$ for every positive integer n, then what can be said about the relation between f and g? Prove your claim.
- Problem 4 on the January 2009 qualifying exam: Prove that if a is an arbitrary complex number, and n is an integer greater than 1, then the polynomial $1 + z + az^n$ has at least one zero in the disk where $|z| \le 2$.
  Hint: The product of the zeroes of a monic polynomial of degree n equals $(-1)^n$ times the constant term.

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- Problem 8 on the January 2009 qualifying exam: Suppose f is holomorphic in the vertical strip where $|\operatorname{Re}(z)| < \pi/4$, and $|f(z)| < 1$ for every z in the strip, and $f(0) = 0$. Prove that $|f(z)| \le |\tan(z)|$ for every z in the strip.

- Problem 5 on the August 2009 qualifying exam: Suppose that f is holomorphic in the disk where $|z| < R$, and $|f(z)| < M$ for every z in this disk. Suppose additionally that there is a point z_0 in the disk for which $f(z_0) = 0$. Prove that
$$|f(z)| \le \frac{MR|z - z_0|}{|R^2 - z\bar{z}_0|} \quad \text{when } |z| < R, \text{ and} \quad |f'(z_0)| \le \frac{MR}{R^2 - |z_0|^2}.$$

- Problem 2 on the January 2010 qualifying exam: Suppose that f has an isolated singularity at the point a, and f'/f has a first-order pole at a. Prove that f has either a pole or a zero at a.

- Problem 3 on the January 2010 qualifying exam: Prove that all the zeroes of the function $\tan(z) - z$ are real.

- Problem 10 on the January 2010 qualifying exam: Prove that if f has isolated singularities at $\pm a$, then $\operatorname{Res}(f, a) = -\operatorname{Res}(f, -a)$ if f is even, and $\operatorname{Res}(f, a) = \operatorname{Res}(f, -a)$ if f is odd.

- Problem 3 on the August 2010 qualifying exam: Calculate the "Fresnel integrals"
$$\int_0^\infty \sin(x^2)\,dx \qquad \text{and} \qquad \int_0^\infty \cos(x^2)\,dx,$$
which play an important role in diffraction theory. (You may assume known that $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$.)

- Problem 10 on the August 2010 qualifying exam: Prove that if f is meromorphic in a complex domain Ω, then so is f'.

- Problem 4 on the January 2011 qualifying exam: Suppose Ω is a connected open subset of $\mathbb{C}$, and $u : \Omega \to \mathbb{R}$ is a nonconstant harmonic function. Prove that $u(\Omega)$, the image of u, is an open subset of $\mathbb{R}$.

- Problem 6 on the January 2011 qualifying exam: Suppose f is a holomorphic function (not necessarily bounded) on $\{ z \in \mathbb{C} : |z| < 1 \}$, the open unit disk, such that $f(0) = 0$. Prove that the infinite series $\sum_{n=1}^{\infty} f(z^n)$ converges uniformly on compact subsets of the open unit disk.