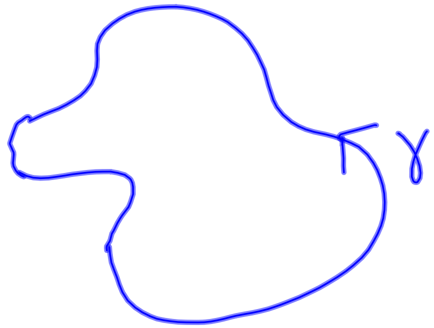
1862

Theorem (Perturbation version of Rouché's theorem)**.** *If f and g are analytic inside a simple closed curve γ, and $|g(z)| < |f(z)|$ when z is* ***on*** *the curve γ, then f and $f \pm g$ have the same number of zeroes inside γ.*

<u>Example</u> $\gamma =$ unit circle



How many zeroes does $z^{10} + 3z^2 + 1$ have inside the unit circle?

Take $f(z) = 3z^2$ $\quad g(z) = z^{10} + 1$.

When $|z| = 1$,

$$|g(z)| \leq 2 < 3 = |f(z)|.$$

Theorem implies that $f(z) + g(z)$ has same # of zeroes inside as $f(z)$: namely, 2.

Theorem (Symmetric version of Rouché's theorem)**.** *If f and g are analytic inside a simple closed curve γ, and $|f(z) + g(z)| < |f(z)| + |g(z)|$ when z is* ***on*** *the curve γ, then f and g have the same number of zeroes inside γ.* [Estermann, 1962]

Rouché's theorem on the qualifying exam

- August 2008 problem 6: If $\alpha > 1$, then the equation $\sin z = e^{\alpha} z^3$ has exactly three solutions in the unit disk.

- August 2009 problem 4: If $a > 0$ and $b > 2$, then the equation $az^3 - z + b = e^{-z}(z + 2)$ has exactly two solutions in the right-hand half-plane. [same as problem 7, August 2014]

- August 2012 problem 7: If $\lambda > 1$, then the equation $e^z - z = \lambda$ has exactly one solution in the left-hand half-plane.