Homotopy proof of perturbation version of Rouché's theorem

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)+tg'(z)}{f(z)+tg(z)}\,dz \qquad (0 \le t \le 1)$$

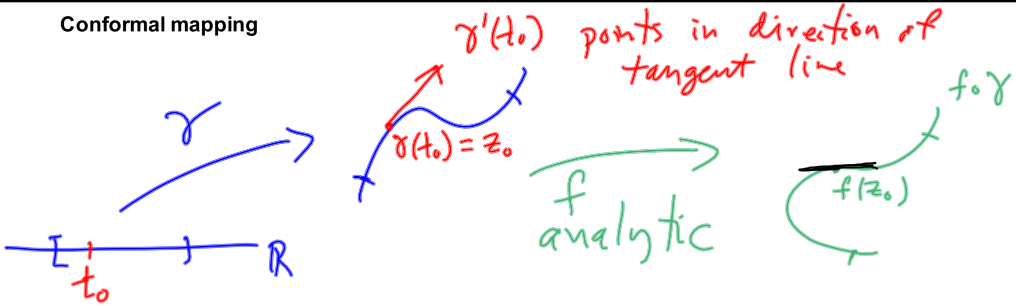
Continuous integer-valued function of t, hence constant.

Hypothesis says $|g(z)| < |f(z)|$ when $z \in \gamma$, hence denominator $\neq 0$.

When $t=0$, integral counts # zeroes of f inside γ.

When $t=1$, integral counts # zeroes of $f+g$.

Conformal mapping



curve γ is a function from interval into $\mathbb{C}$.

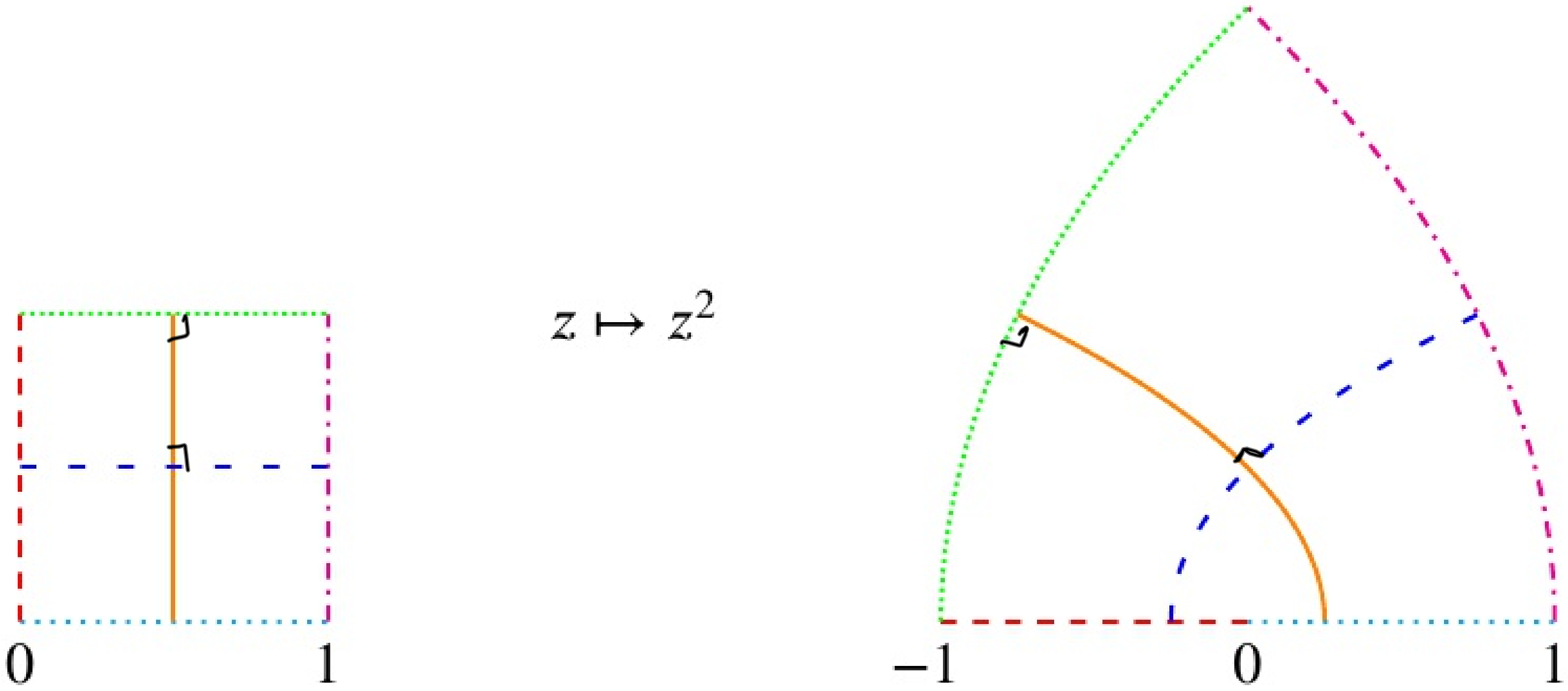
$$(f \circ \gamma)'(t_0) = f'(z_0)\,\gamma'(t_0)$$

The angle of tangent line of image curve has been increased by $\arg(f'(z_0))$

[assuming that $f'(z_0) \neq 0$, so that $\arg f'(z_0)$ makes sense]

If $f' \neq 0$, then f is a conformal mapping

Example: geometry of the squaring function



$\mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$

$(z_1, z_2) \sim (w_1, w_2)$
if $\exists\, \lambda \in \mathbb{C} \setminus \{0\}$ such that
$z_1 = \lambda w_1$ and $z_2 = \lambda w_2$.

$\mathbb{CP}^1$ = set of equivalence classes, $[(z_1, z_2)]$ or $[z_1 : z_2]$.

$\mathbb{C}$ embeds into $\mathbb{CP}^1$ via
$z \longmapsto [z : 1]$

The only equivalence class not obtained this way is $[1 : 0]$.

Alternative embedding of $\mathbb{C}$ into $\mathbb{CP}^1$:
$z \longmapsto [1 : z]$.

How do we pass from one embedding to the other?
$z \longmapsto [z : 1] = [1 : \frac{1}{z}] \longmapsto \frac{1}{z}$