$\log(z)$ should be an inverse of e^z but e^z is not one-to-one.

A <u>branch</u> of $\log(z)$ means

(i) some open subset of $\mathbb{C}$, and

(ii) some analytic function f on that set such that $e^{f(z)} = z$.

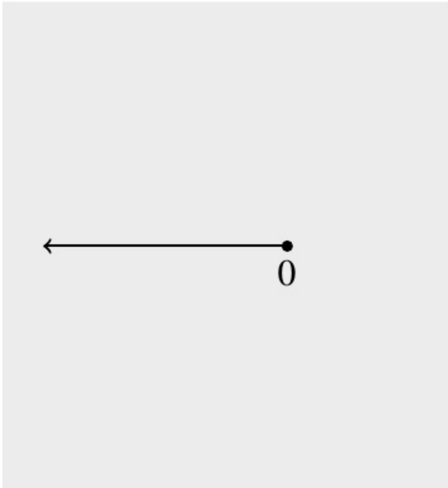
If $f(z)$ exists, then

$$e^{f(z)} = z = re^{i\theta} = e^{\ln r + i\theta}.$$

So the candidate for $\log(z)$ is $\ln|z| + i\arg(z)$.

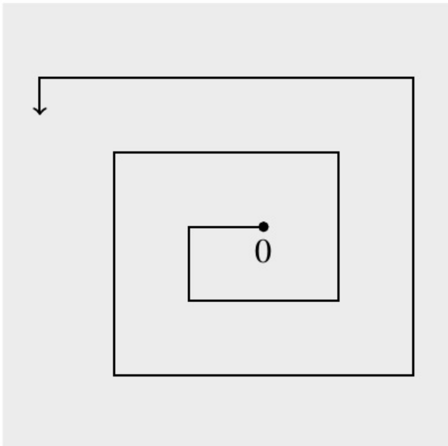
A priori $\arg(z)$ is ambiguous up to addition of $2\pi n$ for some integer n.

Examples of regions where θ can be defined continuously



principal value: $-\pi < \theta < \pi$

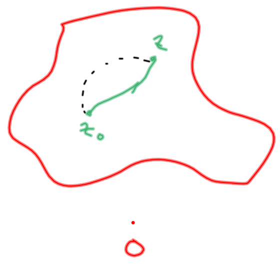
principal branch of $\log(z)$ is $\ln|z| + i\arg(z)$ with $-\pi < \arg(z) < \pi$



θ continuous but unbounded

Proposition If open set G is simply connected, and $0 \notin G$, then a branch of $\log(z)$ can be defined on G.

Why? Define



$$f(z) = \int_{z_0}^{z} \frac{1}{w}\,dw$$

path-independent integral because of version of Cauchy's theorem from last week. And $f'(z) = \frac{1}{z}$.

What do we know about $e^{f(z)}$?

Examine $z \cdot e^{-f(z)}$. Derivative is

$$1 \cdot e^{-f(z)} + z \cdot e^{-f(z)} \cdot (-f'(z))$$

or $e^{-f(z)} - e^{-f(z)} = 0$.

So $z e^{-f(z)} = c$ (constant),

or $z = c \cdot e^{+f(z)}$

$= e^{k} \cdot e^{f(z)}$ for some constant k.

So $k + f(z)$ is a branch of $\log(z)$ in G.

More generally, given an analytic function $f(z)$, and given a domain
is there an analytic function $g(z)$ such that $e^{g(z)} = f(z)$?
[that is, $g = \log f$]

subtle point: $\log f$ does not necessarily mean $\log \circ f$.

If $f(z)$ has zeroes, then no $g(z)$ can exist. If the region is simply connected, then this obstruction is the only one: namely, if f has no zeroes in a simply connected region, then there is g such that $e^g = f$.

Proof Define $g(z)$ to be
$$\int_{z_0}^{z} \frac{f'(w)}{f(w)}\,dw + C$$
and mimic the previous proof.

21. Prove that there is no branch of the logarithm defined on $G = \mathbb{C} - \{0\}$.