Mean-value theorem on real line:

$$f(b) - f(a) = f'(c)(b-a)$$

for some $c$ between $a$ and $b$.

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Counterexample in $\mathbb{C}$:

$$0 = e^{2\pi i} - e^{0} \stackrel{?}{=} \text{(derivative at intermediate point) times } 2\pi i = 0 \; ?$$

**Example** Determine $\dfrac{f'(z)}{f(z)}$ when $f(z) = \dfrac{(z-1)(z-2)}{(z-3)(z-4)}$.

Locally on a small disk not containing 1, 2, 3, or 4, we can define a branch of $\log f(z)$

$$= \log(z-1) + \log(z-2) - \log(z-3) - \log(z-4) + 2\pi i \cdot k$$

for some integer $k$.

Differentiate to get

$$\frac{f'(z)}{f(z)} = \frac{1}{z-1} + \frac{1}{z-2} - \frac{1}{z-3} - \frac{1}{z-4}.$$

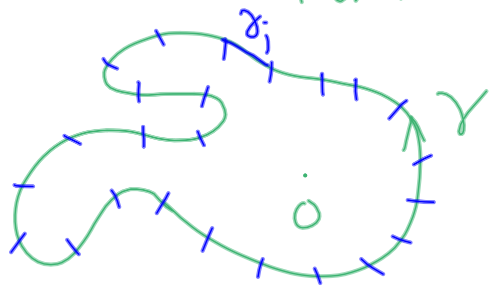
**Theorem.** *Suppose $f$ is analytic in a region $G$ except for some poles, and let $\gamma$ be a simple closed counterclockwise curve that has zero winding number around each point in $\mathbb{C} \setminus G$. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = (\#\ zeroes) - (\#\ poles) \qquad inside\ \gamma.$$

(counted according to multiplicity)

[implicit assumption that no zeroes or poles lie on $\gamma$]

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{w}\,dw = n(\gamma;0) \quad \text{winding number}$$




$$= \frac{1}{2\pi i}\sum_j \int_{\gamma_j} \frac{1}{w}\,dw$$

$$= \frac{1}{2\pi i}\sum_j \log_j w \Big|_{\text{start point of } \gamma_j}^{\text{endpoint of } \gamma_j}$$

well defined!

$$= \frac{1}{2\pi i}\sum_j \Big[\ln|w| + i\arg_j(w)\Big]_{\text{start point}}^{\text{endpoint}}$$

Real part of log gives a telescoping sum that equals 0.

$$\frac{1}{2\pi}\cdot \text{net change in } \arg(w) \text{ around the curve } \gamma.$$

More generally


$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

$$= \frac{1}{2\pi i}\int_{C=f(\gamma)} \frac{1}{w}\,dw \qquad \text{via } w=f(z)$$

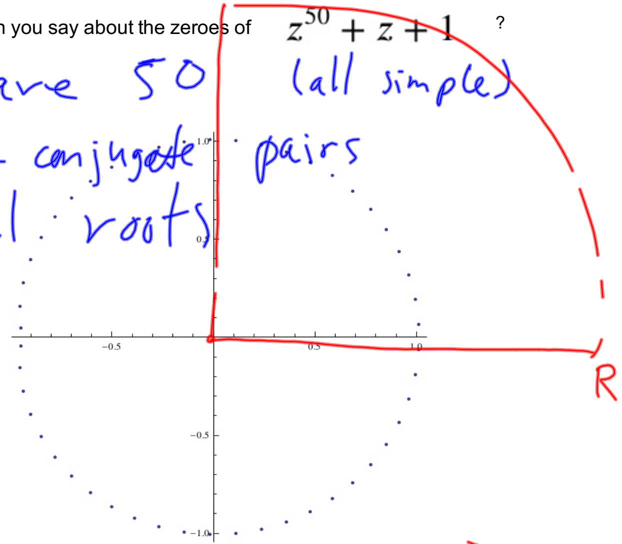
$= \frac{1}{2\pi}$ times change in argument of $f(z)$ as $z$ traverses $\gamma$

**Example.** What can you say about the zeroes of $z^{50} + z + 1$ ?

(a) there are 50 (all simple)

(b) complex-conjugate pairs

(c) no real roots




As $z$ moves from $0$ to $R$, $\arg f(z)$ stays $0$.

As $\theta$ goes from $0$ to $\frac{\pi}{2}$,

$$f(Re^{i\theta}) \approx R^{50} e^{50 i\theta}$$

so $\arg f$ increases by approximately $25\pi$.

As $z = iy$ goes from $y$ big to $y = 0$, $f(z)$ = something with positive imaginary part, so angle changes from $25\pi$ to $24\pi$. Hence there are 12 zeroes in first quadrant.

August 2015 qualifying examination

5. When the variable $z$ is restricted to the first quadrant (where $\operatorname{Re} z > 0$ and $\operatorname{Im} z > 0$), how many zeroes does the polynomial $z^{2015} + 8z^{12} + 1$ have?

January 2013 qualifying examination

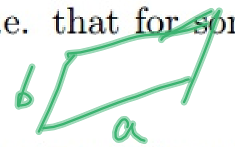
7. Prove that every complex number is in the range of the entire function $e^{3z} + e^{2z}$.

Hint: think about polynomial $w^3 + w^2 - c$.

August 2012 qualifying examination

elliptic functions

3. Let $f$ be a meromorphic function in $\mathbb{C}$. Suppose that $f$ is doubly periodic, i.e. that for some non-zero numbers $a, b \in \mathbb{C}$,




$$f(z) = f(z + a) \text{ and } f(z) = f(z + b)$$

for any $z \in \mathbb{C}$. Consider the parallelogram $P$ with sides $(0, a)$ and $(0, b)$. Assuming that $f$ does not have any zeros or poles on the sides of $P$, prove that the number of zeros of $f$ in $P$ is equal to the number of poles of $f$ in $P$.