Recap

- A function analytic in an annulus can be represented by a Laurent series.

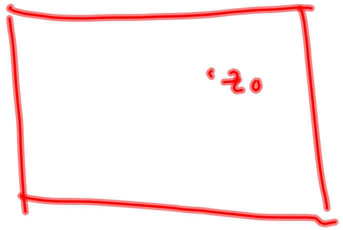
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad c_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

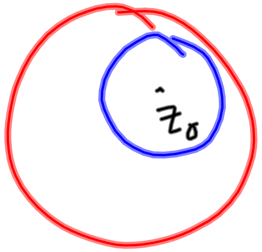
$$C_{-1} = \frac{1}{2\pi i} \int f(z)\, dz$$

- Special case: a function analytic in a disk can be represented by a Taylor series.

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad c_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{f^{(n)}(z_0)}{n!}$$

Residues revisited





If f is analytic except for one point z_0,

then $\displaystyle\int_{\text{boundary}} f(z)\,dz = 2\pi i \cdot (\text{Residue of } f \text{ at } z_0)$

proved so far for first-order singularities.

Now for higher-order singularities, integrate the Laurent series termwise, only the first-order singularity contributes something nonzero.

Example Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx.$

by Cauchy's rectangle theorem,

integral = $2\pi i$ × sum of residues of singularities in upper half-plane.

$$\frac{1}{(z^2+1)^2}$$ want residue at $z = i$

$$= \frac{(z+i)^{-2}}{(z-i)^2} = \frac{c_{-2}}{(z-i)^2} + \frac{c_{-1}}{(z-i)} + c_0 + c_1(z-i) + \cdots$$

c_{-1} = coeff. of $(z-i)^1$ in Taylor series of $(z+i)^{-2}$ centered at i.

$$\frac{(z+i)^{-2}}{(z-i)^2} = \frac{a_0 + a_1(z-i) + a_2(z-i)^2 + \cdots}{(z-i)^2}$$

$$c_{-1} = a_1 = \frac{d}{dz}(z+i)^{-2}\Big|_{z=i}$$

by the formula for Taylor coefficients

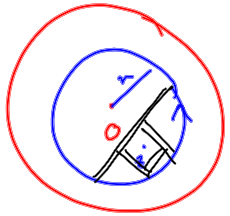
$$= -2(z+i)^{-3}\Big|_{z=i}$$

$$= \frac{1}{4i}$$

So integral = $\dfrac{2\pi i}{4i} = \dfrac{\pi}{2}$

Riemann's theorem on removable singularities

If f is analytic in a punctured disk and *bounded*, then the singularity at the puncture is a removable singularity, that is, f can be defined at the puncture so as to become analytic in the whole disk.

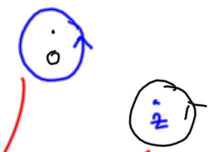


The only candidate to extend f is the Cauchy integral

$$\frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)}{w-z}\,dw$$

Does this integral recover the original $f(z)$ when $z \neq 0$?

Add and subtract integrals over line segments to break up the integral into



plus some integrals that are equal to 0 because they surround no singularities.

this integral $= f(z)$ by Cauchy

this integral has absolute value $\leq$ (upper bound for $|f|$) times (length of integration path) $\to 0$

August 2015 qualifying examination

4. When n is an integer, the Bessel function $J_n(z)$ can be defined to be the coefficient of t^n in the Laurent series about the origin of

$$\exp\left(\frac{1}{2}z\left(t-\frac{1}{t}\right)\right)$$

(series with respect to the variable t). Use this definition to show that $J_{-n}(z) = (-1)^n J_n(z)$.

January 2015 qualifying examination

2. There is a holomorphic function f such that $f(z)e^{f(z)} = z$ for every z in a neighborhood of the origin. Find the first three nonzero terms in the Maclaurin series of f.

3. Prove that $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{\cos\theta}\, d\theta = \sum_{n=0}^{\infty} \frac{1}{(n!\, 2^n)^2}$.