Classification of isolated singularities

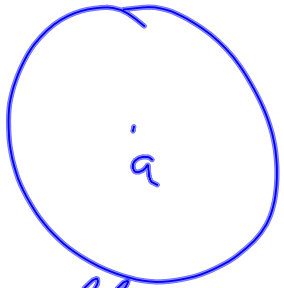
1. Removable singularity: f is bounded near the puncture. Equivalently, by Riemann's theorem, f has a finite limit at the puncture. Equivalently, the Laurent series has no terms with negative exponents.

2. Pole: $|f| \to \infty$ at the puncture. Equivalently, $1/f$ has a removable singularity with value 0 at the puncture. Equivalently, the Laurent series has some, but finitely many, terms with negative exponents.

3. Essential singularity: $|f|$ has neither a finite nor an infinite limit at the puncture. Equivalently, the Laurent series has infinitely many terms with negative exponents.

$e^{\frac{1}{z}}$ has essential singularity at 0

Theorem (Casorati–Weierstrass)**.** *The range of an analytic function in a punctured neighborhood of an essential singularity is dense in* $\mathbb{C}$.



Proof Suppose not.

Then there is some disk $B(c; r)$ disjoint from the range.

That is, $|f(z) - c| \geq r$ for all z in the punctured domain of f.

Then $\dfrac{1}{f(z) - c}$ is bounded, hence has removable singularity at $z = a$.

$\displaystyle\lim_{z \to a} \frac{1}{f(z) - c}$ exists.

Case 1: this limit is 0. Then f has a pole at $z = a$.

Case 2: This limit is not 0. Then f has removable singularity. Both cases contradict the hypothesis.

Theorem (Picard's "great theorem" or "big theorem")**.** *In every punctured neighborhood of an essential singularity, an analytic function assumes every complex value—with one possible exception—infinitely often.*

2. a) Find and classify all isolated singularities of

$$f(z) = \frac{z^2(z-\pi)}{\sin^2 z} \quad \text{and} \quad g(z) = (z^2-1)\cos\frac{1}{z-1}.$$

b) Find the residue of f at $z = 2\pi$ and the residue of g at $z = 1$.