Recap of Cauchy's rectangle theorems

1. If f is analytic in an open set, and R is a rectangle contained in the set, then

$$\int_{\partial R} f(z)\, dz = 0.$$

2. If f is analytic in an open set, except for some isolated singularities, and R is a rectangle contained in the set, then

$$\int_{\partial R} f(z)\, dz = 2\pi i \times (\mathit{sum\ of\ residues\ at\ the\ singular\ points}).$$

(proved so far only for first-order singularities)

Cauchy's integral formula for rectangles

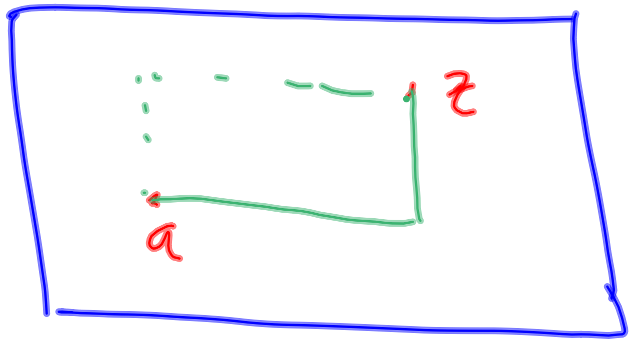
If f is analytic in an open set, and R is a rectangle contained in the set, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(w)}{w - z}\, dw \qquad \text{(when } z \text{ is inside } R\text{)}$$

Corollary from last time

An analytic function f in a rectangle has an analytic antiderivative: namely,

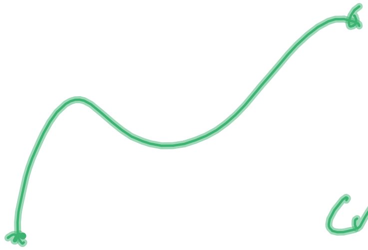
$$F(z) := \int_a^z f(w)\,dw$$



Line integrals (path integrals)

curve

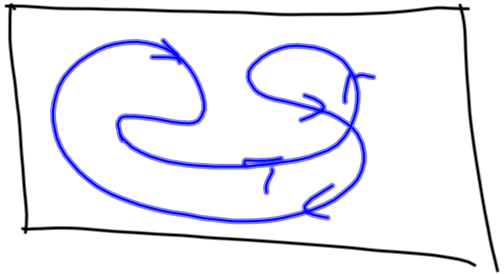
$\gamma : [0,1] \to \mathbb{C}$



continuously differentiable

$\int_\gamma f(z)\,dz$ by definition means $\int_0^1 f(\gamma(t))\gamma'(t)\,dt$.

Theorem (Cauchy's integral theorem generalized to curves). *If f is an analytic function inside a rectangle, and γ is a continuously differentiable closed curve lying inside the rectangle, then* $\int_\gamma f(z)\,dz = 0$.



Proof

$$\int_0^1 f(\gamma(t))\,\gamma'(t)\,dt = ?$$

Suppose $F'(z) = f(z)$.

Then $\frac{d}{dt} F(\gamma(t)) \underset{\text{chain rule}}{=} F'(\gamma(t)) \cdot \gamma'(t)$

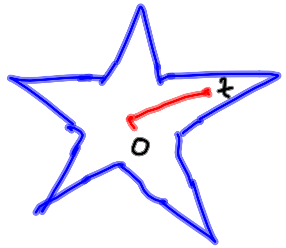
$= f(\gamma(t))\,\gamma'(t)$

So $\int_0^1 f(\gamma(t))\gamma'(t)\,dt = F(\gamma(t))\Big|_0^1 = F(\gamma(1)) - F(\gamma(0))$

$= 0$

since curve is closed.

Generalization to starshaped regions



Theorem. *An analytic function in a star-shaped region has an anti-derivative.*

$$F(z) = \int_0^z f(w)\,dw$$

integrate along a line segment from the base point to z.