Cauchy-Riemann equations for $f = u + iv$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Exercise from page 44

14. Suppose $f\colon G \to \mathbb{C}$ is analytic and that G is connected. Show that if $f(z)$ is real for all z in G then f is constant.

$v \equiv 0$

In Euclidean space, connected $\iff$ path connected.

Wirtinger's notation, 1927

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial z}(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy) = 1$$

$$\frac{\partial}{\partial \overline{z}}(\overline{z}) = 1 \qquad \frac{\partial}{\partial \overline{z}}(z) = 0 = \frac{\partial}{\partial z}(\overline{z})$$

Cauchy–Riemann equations rewritten: $\dfrac{\partial f}{\partial \overline{z}} = 0$

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) =$$

$$\frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Laplace's equation (harmonic functions)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or

$$\frac{\partial^2 u}{\partial \bar{z}\, \partial z} = 0$$

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y}$$

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Connections between harmonic functions and analytic functions

1. The real part of an analytic function is harmonic.

2. The converse is true *locally.* (preview of coming attraction)

Some elementary functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$e^i = 1 + \frac{i}{1} + \frac{i^2}{2!} + \frac{i^3}{3!} + \frac{i^4}{4!} + \cdots$$

$$= 1 + i - \frac{1}{2!} - \frac{i}{3!} + \frac{1}{4!} - \cdots$$

$$= \left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots\right) + i\left(1 - \frac{1}{3!} + \frac{1}{5!} - \cdots\right)$$

$$= \cos(1) + i\sin(1)$$

More generally,

$$\boxed{e^{iz} = \cos(z) + i\sin(z)}$$

so

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

Persistence of functional relations

$(\sin x)^2 + (\cos x)^2 = 1$ for every real number x.

Is $(\sin z)^2 + (\cos z)^2 = 1$ for every complex number z?

Yes: if two power series with infinite radius of convergence match on the real axis, then the coefficients match, so the series match everywhere in $\mathbb{C}$.

$|\sin(x)| \leq 1$ for every real number x.
Is $|\sin(z)| \leq 1$ for every complex number z?

No!

Exercises from page 44

6. Describe the following sets: $\{z\colon e^z = i\}$, $\{z\colon e^z = -1\}$, $\{z\colon e^z = -i\}$, $\{z\colon \cos z = 0\}$, $\{z\colon \sin z = 0\}$.

19. Let G be a region and define $G^* = \{z\colon \bar{z} \in G\}$. If $f\colon G \to \mathbb{C}$ is analytic prove that $f^*\colon G^* \to C$, defined by $f^*(z) = \overline{f(\bar{z})}$, is also analytic.