Comparison of $\mathbb{C}$ and $\mathbb{R}$

$\mathbb{C}$ is algebraically closed.

$\mathbb{R}$ has an order relation.

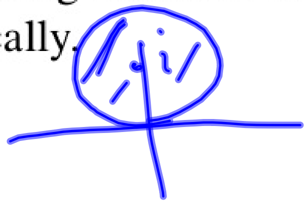
Comparison of $\mathbb{C}$ and $\mathbb{R}^2$

Isomorphic as real vector spaces.

Exercise. Show that a real-linear transformation of $\mathbb{R}^2$, represented by a real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, corresponds to a complex-linear transformation of $\mathbb{C}$ if and only if $a = d$ and $b = -c$.

Exercise. Each of the following relations in z defines a certain subset of the complex plane. Describe each set geometrically.

1. $|z - i| < 1$



open disk

2. $\mathrm{Re}[(1 + i)z] > 0$

open half-plane

3. $|z| + |z - 1| = 2$

ellipse!

4. $\mathrm{Re}(z) = |z - 1|$

parabola with focus 1 directrix y-axis

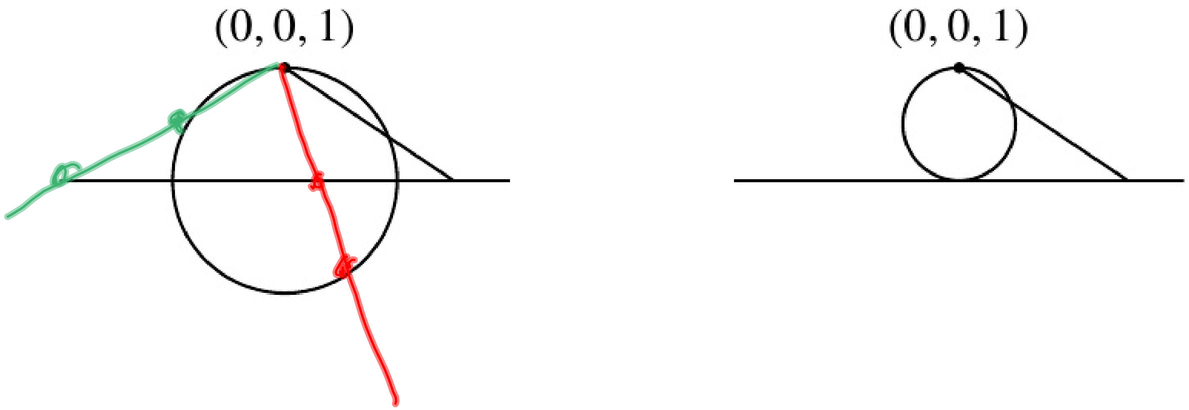
5. $|z - i| < |z + i|$

open upper half-plane

6. $\mathrm{Re}\,\frac{z-1}{z+1} < 1$

half-plane

Two models of stereographic projection



Exercise. Derive the following formulas for the model of stereographic projection that uses a sphere of diameter 1 tangent to the complex plane at the origin (a different model from the one in the textbook).

If $x + iy$ (or z) is a point in the complex plane, and (x_1, x_2, x_3) is the corresponding point on the sphere, then

$$x = \frac{x_1}{1 - x_3} \qquad \text{and} \qquad y = \frac{x_2}{1 - x_3};$$

also

$$x_1 = \frac{x}{|z|^2 + 1} \qquad \text{and} \qquad x_2 = \frac{y}{|z|^2 + 1} \qquad \text{and} \qquad x_3 = \frac{|z|^2}{|z|^2 + 1}.$$

Moreover, the spherical distance d for this model has the following properties:

$$d(z, z') = \frac{|z - z'|}{\sqrt{|z|^2 + 1}\,\sqrt{|z'|^2 + 1}} \quad \text{and} \quad d(z, \infty) = \frac{1}{\sqrt{|z|^2 + 1}}.$$