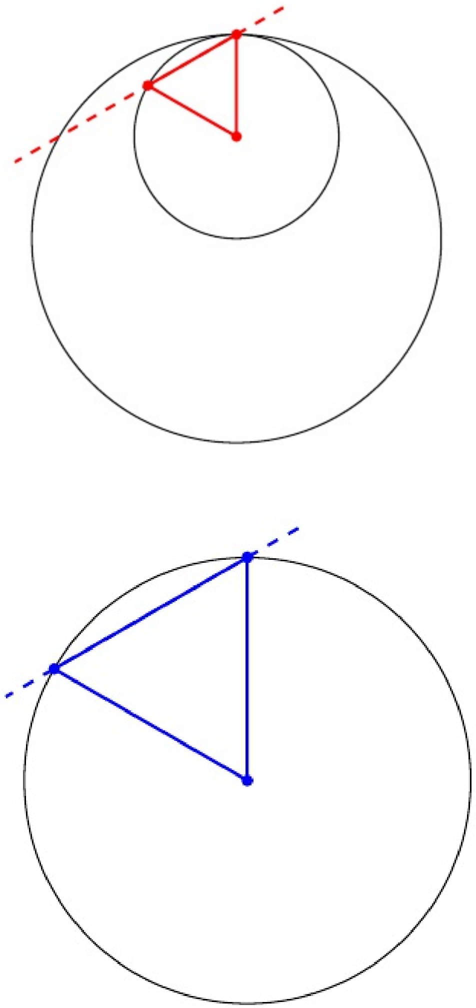
two stereographic projections
with the same north pole
are compatible



Theorem (Weierstrass approximation theorem). *Every continuous real-valued function* $f(x)$ *on the closed interval* $[-1, 1]$ *can be obtained as the uniform limit of polynomials in* x.

Uniform means

$$\max_{-1 \le x \le 1} |p_n(x) - f(x)| \longrightarrow 0 \quad \text{as } n \to \infty$$

Question. Can every continuous complex-valued function $f(z)$ on $\{ z \in \mathbb{C} : |z| \leq 1 \}$, the closed unit disk, be obtained as the uniform limit of polynomials in z?

No, the function $\bar{z}$ cannot be obtained.

Proof is coming soon.

Exercise. Prove Cauchy's logarithm test for infinite series of positive terms. Namely, suppose that $a_n > 0$ for every natural number n, and suppose that the limit

$$\lim_{n\to\infty} \frac{\log a_n}{\log(1/n)} = \lim \frac{\log\left(\frac{1}{a_n}\right)}{\log(n)}$$

exists and equals h. Show that if $h > 1$, then $\sum_{n=1}^{\infty} a_n$ converges, and if $h < 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Warning: $\log(1/n)$ is negative!

Convergence tests for $\sum_{n=1}^{\infty} a_n$

- ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ < 1 series converges
  > 1 series diverges
  $= 1$ inconclusive

- root test $\limsup_{n\to\infty} |a_n|^{1/n}$ same decision rule as above

Cauchy condensation test
for positive decreasing terms
$0 \leq a_{n+1} \leq a_n$.

$\sum_{n=1}^{\infty} a_n$ and $\sum 2^n a_{2^n}$

either both converge
or both diverge.