Examination 2

1. State the following theorems: Liouville's theorem about entire functions, Morera's theorem, and the Casorati–Weierstrass theorem.

   Solution. See Theorems IV.3.4, IV.5.10, and V.1.21 in the textbook.

2. Give an example of an open set, an analytic function f defined on the set, and two paths γ_1 and γ_2 in the set having the same endpoints [in other words, $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$] such that $\int_{\gamma_1} f(z)\,dz \neq \int_{\gamma_2} f(z)\,dz$.

   Solution. One example is the open set $\mathbb{C} \setminus \{0\}$, the function $1/z$, and the paths with parametrizations $\exp(\pi i t)$ and $\exp(-\pi i t)$. The paths are the top half and the bottom half of the unit circle, starting at 1 and ending at -1. The integrals over the two paths are equal to πi and $-\pi i$.

   Another example using the same domain and function is the pair of closed paths $e^{2\pi i t}$ and $e^{4\pi i t}$ (the unit circle starting at 1 and ending at 1, traversed either once or twice). The integral over the first path equals $2\pi i$, and the integral over the second path equals $4\pi i$. (This example is contained in Exercise 9 on page 67 of the textbook, a homework assignment from October 6.)

3. Determine (with proof) the maximum value and the minimum value of the real-valued expression $\left|z^2 - 1\right|$ when $|z| \le 1$.

   Solution. The smallest possible absolute value is 0, and evidently this minimum is attained when $z = \pm 1$. Here are two arguments for determining the maximum value.

   By the triangle inequality, $\left|z^2 - 1\right| \le \left|z^2\right| + 1 = |z|^2 + 1 \le 1 + 1$ when $|z| \le 1$. This upper bound 2 is attained when $z = \pm i$, hence is the maximum.

   Alternatively, argue by the maximum principle that the maximum must be attained on the boundary, where $z = e^{i\theta}$. Then

   $$\left|z^2 - 1\right| = \left|e^{2i\theta} - 1\right| = \left|e^{i\theta}\left(e^{i\theta} - e^{-i\theta}\right)\right| = 2|\sin\theta|.$$

   Evidently the maximum value of the right-hand side equals 2 (attained when $\theta = \pi/2$ and when $\theta = 3\pi/2$).

Examination 2

4. Suppose f is analytic in $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$, the punctured unit disk. If f has a removable singularity at the origin, then what can you say about the singularity of $1/f$, the reciprocal function?

Solution. By hypothesis, $\lim_{z\to 0} f(z)$ exists. If this limit is different from 0, then $1/f$ has a finite limit near the origin, whence $1/f$ has a removable singularity at the origin. If instead the limit of f is 0, then there are two subcases.

(a) If f has no other zero in a neighborhood of the origin, then $1/f$ is analytic in a punctured neighborhood of the origin and has limit ∞ at the origin, so $1/f$ has a pole at the origin. (The order of the pole is equal to the order of the zero of f.)

(b) If f has another zero in every neighborhood of the origin, then f is identically zero by the identity principle. In that case, $1/f$ is everywhere undefined, so $1/f$ has a nonisolated singularity at the origin.

Remark If f has a pole at the origin instead of a removable singularity, then $|f| \to \infty$ at the origin, so $1/f \to 0$, whence $1/f$ has a removable singularity. Moreover, the order of the zero of $1/f$ at the origin equals the order of the pole of f at the origin.

Finally, consider the case that f has an essential singularity at the origin. If f has a zero in every punctured neighborhood of the origin, then $1/f$ has a nonisolated singularity at the origin. This situation is typical, in view of Picard's great theorem. If 0 happens to be a Picard exceptional value, so that there is some punctured neighborhood of the origin in which f is nowhere equal to 0, then $1/f$ is well defined in a punctured neighborhood of the origin. The isolated singularity of $1/f$ must then be essential, for otherwise one of the previous cases would be contradicted.

5. Does there exist an entire function f such that $f(n) = n \cdot (-1)^n$ for every natural number n?

Solution. Yes. The canonical example of such a function is $z\cos(\pi z)$.

Examination 2

Remarks The function $f(z) - z$ has a zero at each even natural number, and the function $f(z) + z$ has a zero at each odd natural number. These infinite sequences of zeroes do not violate the identity theorem, for the zeroes have no accumulation point in the complex plane. If you work in the extended complex numbers, then the zeroes accumulate at the point at infinity; there is still no violation of the identity theorem, for the function is not analytic at infinity—indeed, there is an essential singularity at infinity.

The function f is not unique: another example is $ze^{\sin(\pi z)}\cos(\pi z)$. But the function $z\cos(\pi z)$ is the "smallest" example in the following sense. If c is an arbitrary real number strictly less than π, then there is no solution f for which the function $|f(z)|e^{-c|z|}$ is bounded. This conclusion follows from a theorem of R. C. Buck in *Duke Mathematical Journal* **13** (1946) 541–559.

Notice that if the target value for $f(n)$ were slightly perturbed, then there would be no obvious way to write down a solution. One of the tasks in Math 618 next semester will be to develop some constructive methods for producing analytic functions having prescribed properties.

6. Determine the residue of the rational function $\dfrac{1}{(z^2-1)^5}$ at the point where $z = 1$.

Solution. View the function as $\dfrac{(z+1)^{-5}}{(z-1)^5}$. The required value is the coefficient of $(z-1)^4$ in the Taylor expansion of the numerator about the point 1. This coefficient equals

$$\frac{1}{4!} \cdot \frac{d^4}{dz^4}(z+1)^{-5} \bigg|_{z=1}.$$

This value works out to be

$$\frac{(-5)(-6)(-7)(-8)}{4! \cdot 2^9}, \qquad \text{or} \qquad \frac{35}{256}.$$