Examination 2

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Give an example of a closed curve γ such that the integrals $\int_\gamma \frac{8}{z-11}\,dz$ and $\int_\gamma \frac{11}{z-8}\,dz$ are well defined, equal, and nonzero.

2. Suppose G is a simply connected open set, and f is an analytic function on G without zeros. You know a theorem stating that there exists a logarithm of f, that is, an analytic function g such that $e^{g(z)} = f(z)$ when $z \in G$.
   (a) If f is injective, must g be injective?
   (b) If g is injective, must f be injective?

   Explain your reasoning.

3. How many zeros does the function $z^{2018} + 11z^8 + e^z$ have in the annulus where $1 < |z| < 2$? Explain how you know.
   (As usual, zeros are to be counted according to multiplicity.)

4. In some disk with center $11 + 8i$, the function $\frac{1}{\cos(z)}$ can be represented by a Taylor series $\sum_{n=0}^{\infty} c_n(z - 11 - 8i)^n$. You know a theorem guaranteeing the existence of a radius R such that this series converges when $|z - 11 - 8i| < R$ and diverges when $|z - 11 - 8i| > R$. Determine the greatest integer less than or equal to R.

5. Prove there is *no* analytic function f on the disk $\{ z \in \mathbb{C} : |z| < 2018 \}$ such that

$$f(1/n) = \begin{cases} 1/n^8, & \text{when } n \text{ is an even natural number,} \\ 1/n^{11}, & \text{when } n \text{ is an odd natural number.} \end{cases}$$

6. (a) A student makes an error by claiming that if f is an analytic function on a connected open set G, then $\int_\gamma f(z)\,dz = 0$ for every simple closed smooth curve γ in G. Show that the claim is false by giving a counterexample. (You get to choose G and f and γ.)
   (b) The student makes more error by claiming that if g is a continuous function on a connected open set G, and if there exists a simple closed smooth curve γ in G for which $\int_\gamma g(z)\,dz = 0$, then g is analytic on G. Show that this claim is false too by giving a counterexample.