Theory of Functions of a Complex Variable I

Instructions Solve **six** of the following seven problems. Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "determine" or that asks a question requires an explanation to support the answer. Please explain your reasoning in complete sentences.

1. Suppose that a and b are positive real numbers, and $\gamma(t) = a\cos(t) + ib\sin(t)$, where $0 \le t \le 2\pi$. (The path γ is an ellipse.) Show that

$$\int_0^{2\pi} \frac{1}{a^2\cos^2(t) + b^2\sin^2(t)}\,dt = \frac{2\pi}{ab}$$

by evaluating the path integral $\int_\gamma (1/z)\,dz$ in two different ways: by applying Cauchy's integral formula and by using the explicit parametrization of γ.

2. If f and g are two entire functions such that $e^{f(z)} = e^{g(z)}$ for all z, then what can you deduce about the relationship between the functions f and g?

3. Determine the maximum value of $\operatorname{Re}(z^3)$ when z lies in the unit square $[0,1]\times[0,1]$.

4. Prove that if f is an entire function such that $|f(z)| > 1$ for all z, then f must be a constant function.

5. Suppose f is analytic in the disk where $|z| < 10$, the modulus $|f(z)|$ is bounded above by 28 for all z in this disk, and $f(0) = 0$. How large can $|f(5)|$ be?

6. Show that there does *not* exist an analytic function f in the disk $D(0,2)$ such that

$$f(1/n) = (-1)^n/n \qquad \text{for every natural number } n.$$

7. Suppose f is a nonconstant analytic function in the unit disk $D(0,1)$, not necessarily one-to-one, and the image of f is a region Ω. Must Ω be a simply connected region?