Welcome to Math $(16^2 + 19^2)$

Theory of Functions of a Complex Variable I

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August 28, 2018

An excerpt from Cauchy's 1825 memoir

$$\int_0^\infty x^{a-1} \sin\left(\frac{a\pi}{2} - bx\right) \frac{r\,\mathrm{d}x}{r^2 - x^2} = \frac{\pi}{2} r^{a-1} \cos\left(\frac{a\pi}{2} - br\right), \tag{23}$$

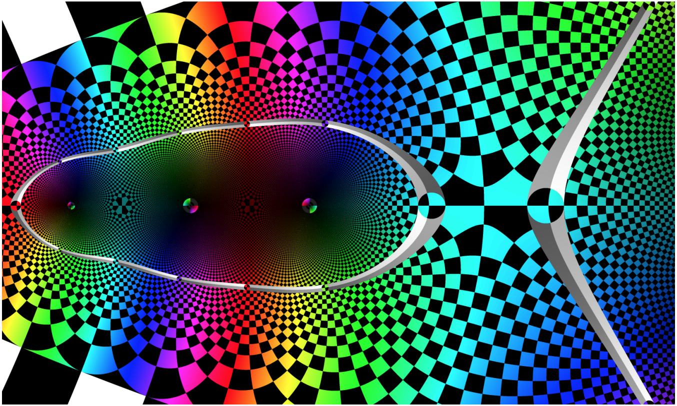
$$\int_0^\infty e^{a\cos bx} \sin(a \sin bx) \frac{\mathrm{d}x}{x} = \frac{\pi}{2} e^a, \tag{24}$$

$$\int_0^\infty e^{a\cos bx} \cos(a \sin bx) \frac{r\,\mathrm{d}x}{x^2 + r^2} = \int_0^\infty e^{a\cos bx} \sin(a \sin bx) \frac{x\,\mathrm{d}x}{x^2 + r^2} = \frac{\pi}{2} e^{ae^{-br}}, \tag{25}$$

$$\int_0^\infty x^{a-1} e^{\cos bx} \sin\left(\frac{a\pi}{2} - \sin bx\right) \frac{r\,\mathrm{d}x}{x^2 + r^2} = \frac{\pi}{2} r^{a-1} e^{e^{-br}}. \tag{26}$$

By the end of the semester, you will have the tools to evaluate these integrals. (One of Cauchy's answers is wrong.)

An example from David Bau's conformal map viewer



By the end of the semester, you will be able to explain what the program at http://davidbau.com/conformal/ does.

The Riemann mapping theorem

By the end of the semester, you will be able to prove the following.

Theorem

A proper planar region is topologically equivalent to a disk if and only if it is analytically equivalent to a disk.

(homeomorphic to a disk $\iff$ biholomorphic to a disk)

What are the complex numbers?

- The everyday working definition: Expressions of the form $a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and i is a formal symbol with the property that $i^2 = -1$.
- Hamilton's definition in terms of real numbers: Ordered pairs (a, b) with vector addition and with the special multiplication law that $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.
- Cauchy's algebraic definition: $\mathbb{R}[x]/(x^2 + 1)$.
- Matrix representation: Expressions $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, with matrix addition and matrix multiplication.

What structures does $\mathbb{C}$ support?

- Algebraic structure: $\mathbb{C}$ is a field, the algebraic closure of $\mathbb{R}$.
- Vector space structure: $\mathbb{C}$ is a two-dimensional real vector space and a one-dimensional complex vector space.
- Topological structure: A subset of $\mathbb{C}$ is *open* iff each point of the subset is the center of some disk contained in the subset.
- $\mathbb{C}$ is *not* an *ordered* field.
- Metric structure: The topology of $\mathbb{C}$ is induced by a distance function. Namely, $d(z, w) = |z - w|$, where $|a + bi|$ means $\sqrt{a^2 + b^2}$.
- Geometric structure: $\mathbb{C}$, viewed as $\mathbb{R}^2$, supports analytic geometry.

Exercise on geometry in $\mathbb{C}$

Each of the following expressions in the complex variable z defines a certain subset of the complex plane. Describe each set geometrically (for example, a line or a circle or a half-plane).

1. $|z - i| < 1$
2. $\mathrm{Re}[(1 + i)z] > 0$
3. $|z| + |z - 1| = 2$
4. $\mathrm{Re}(z) = |z - 1|$
5. $|z - i| < |z + i|$
6. $\mathrm{Re}\,\frac{z-i}{z+i} < 1$

Assignment due next class

- Read sections 1, 2, and 3 in Chapter I of the textbook.