Recap

- The complex numbers are a field, but not an ordered field.
- The fundamental theorem of algebra says that the field $\mathbb{C}$ is algebraically closed.
- The complex numbers form a metric space, the standard Euclidean distance $d(z, w)$ being $|z - w|$.

Exercise on limits

Discuss the following limits.

1. $\lim_{z \to i} z^n \quad (\text{where } n \in \mathbb{N})$
2. $\lim_{n \to \infty} z^n \quad (\text{where } z \in \mathbb{C})$

Answer to exercise

1. $$\lim_{z \to i} z^n = \begin{cases} 1, & \text{if } n \equiv 0 \mod 4 \\ i, & \text{if } n \equiv 1 \mod 4 \\ -1, & \text{if } n \equiv 2 \mod 4 \\ -i, & \text{if } n \equiv 3 \mod 4 \end{cases}$$

2. $$\lim_{n \to \infty} z^n = \begin{cases} 0, & \text{if } |z| < 1 \\ 1, & \text{if } z = 1 \\ \text{does not exist}, & \text{if } |z| = 1 \text{ but } z \neq 1 \\ \infty, & \text{if } |z| > 1 \end{cases}$$

Reminders on limit proofs

To prove in detail that $\lim_{z \to i} z^n = i^n$, what needs to be shown is that for every positive ε, there exists a positive δ with the following property: if $|z - i| < \delta$, then $|z^n - i^n| < \varepsilon$.

To execute the proof, observe that

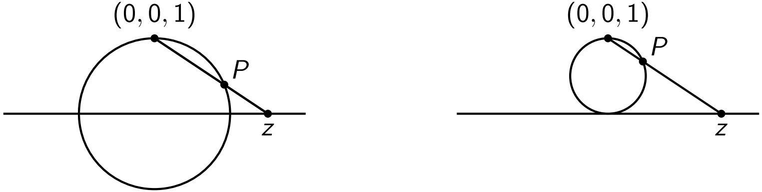
$$z^n - i^n = (z - i)(z^{n-1} + z^{n-2} i + \cdots + z\, i^{n-2} + i^{n-1}),$$

so the triangle inequality implies that if $|z| \leq 2$, then

$$|z^n - i^n| \leq |z - i| \, (2^{n-1} + 2^{n-2} + \cdots + 1) \leq |z - i| \, 2^n.$$

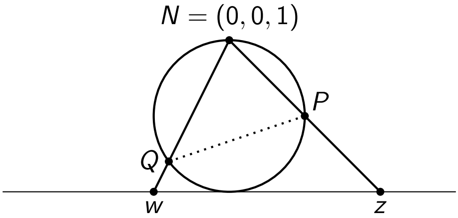
Therefore, if $\delta = \min(1, \varepsilon/2^n)$, then the required property holds: namely, if $|z - i| < \delta$, then $|z^n - i^n| < \varepsilon$.

$\mathbb{C}_\infty$ and stereographic projection



Assignment due next class

- Read the rest of Chapter I.
- For the tangent-sphere model of stereographic projection, show that the spherical distance between complex numbers z and w equals $\dfrac{|z-w|}{\sqrt{1+|z|^2}\,\sqrt{1+|w|^2}}$.



Hint: To minimize computation, show that triangles NPQ and Nwz are similar.