Limits of functions

Suppose S is a subset of $\mathbb{C}$, and $f\colon S \to \mathbb{C}$ is a function, and p is a limit of a sequence of points of the set S. The statement "$\lim_{z\to p} f(z) = c$" means either of the following equivalent properties.

1. For every positive ε there exists a positive δ such that $|f(z) - c| < \varepsilon$ whenever $z \in S$ and $0 < |z - p| < \delta$.
2. Whenever $\{z_n\}_{n=1}^{\infty}$ is a sequence of points of $S \setminus \{p\}$ that converges to p, the image sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to c.

Exercise

How should these properties be rephrased

(a) when $p = \infty$?

(b) when $c = \infty$?

(c) when $p = \infty$ and $c = \infty$?

Continuous functions

Suppose S is a subset of $\mathbb{C}$, and $f\colon S \to \mathbb{C}$ is a function, and p is a point of the set S. The statement "f is continuous at p" means any one of the following equivalent properties.

1. For every positive ε there exists a positive δ such that $|f(z) - f(p)| < \varepsilon$ whenever $z \in S$ and $|z - p| < \delta$.
2. Whenever $\{z_n\}_{n=1}^{\infty}$ is a sequence of points of S that converges to p, the image sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to $f(p)$.
3. Whenever B is a disk centered at $f(p)$, the inverse image $f^{-1}(B)$ contains the intersection of S with a disk centered at p.

The statement "f is continuous on S" means that f is continuous at p for every point p in S.

Exercise (not to hand in)

Convince yourself that you can prove that continuity is preserved by forming

1. sums of functions
2. products of functions
3. compositions of functions

Derivatives

Suppose G is an *open* subset of $\mathbb{C}$, and $f\colon G \to \mathbb{C}$ is a function, and p is a point of the set G. The statement "f is (complex) differentiable at p" means any one of the following equivalent properties.

1. $\displaystyle\lim_{z\to p} \frac{f(z)-f(p)}{z-p}$ exists (as a complex number, not ∞).
2. There exists a complex-linear function $\ell\colon \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{z\to p} \frac{f(z)-f(p)-\ell(z-p)}{z-p} = 0.$$

3. There exists a function $\widetilde{f}\colon G \to \mathbb{C}$, continuous at p, such that $f(z) - f(p) = \widetilde{f}(z)(z-p)$.

The *derivative* $f'(p)$ means the value of the limit in property 1, and $\ell(z)/z$ in property 2, and $\widetilde{f}(p)$ in property 3.

Assignment due next class

- Show that a real-linear transformation of $\mathbb{R}^2$, represented by a real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, corresponds to a complex-linear transformation of $\mathbb{C}$ if and only if $a = d$ and $b = -c$.