# Recap: Cauchy–Riemann equations

When $G$ is an open subset of $\mathbb{C}$, and $f\colon G \to \mathbb{C}$ is a function expressed as $u(x, y) + iv(x, y)$, the Cauchy–Riemann equations say that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

# Some theorems about the Cauchy–Riemann equations

Suppose $G$ is an open subset of $\mathbb{C}$, and $f\colon G \to \mathbb{C}$ is a function, and $f(z) = u(x, y) + iv(x, y)$, where $z = x + yi$.

1. If $u$ and $v$ have continuous first-order (real) partial derivatives, then $f$ is analytic on $G$ iff the Cauchy–Riemann equations hold on $G$. [Theorem 2.29 in Chapter III]
2. If $f$ is analytic, then $u$ and $v$ do have continuous real partial derivatives [see Corollary 2.12 in Chapter IV], and the Cauchy–Riemann equations do hold.
3. If $f$ is continuous, and the first-order partial derivatives of $u$ and $v$ exist (not necessarily continuous) and satisfy the Cauchy–Riemann equations in $G$, then $f$ is analytic on $G$. [Looman–Menshov theorem, not in the book]
4. If $f$ is continuous, and the Cauchy–Riemann equations hold in the sense of distributions (generalized functions), then $f$ is analytic. [not in the book; standard knowledge in PDE]

# A deduction about the range of an analytic function

Suppose $G$ is a nonvoid connected open set, and $f\colon G \to \mathbb{C}$ is an analytic function. If the range of $f$ is a subset of $\mathbb{R}$, then $f$ must be a constant function. [Exercise 14 in §2 of Chapter III]
Why?

Write $f$ as $u + iv$. The hypothesis says that $v$ is the zero function. The Cauchy–Riemann equations then imply that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are identically equal to zero. Therefore $u$ is constant along horizontal lines and also is constant along vertical lines. So $u$ is a constant function. Hence $f$ is a constant function.

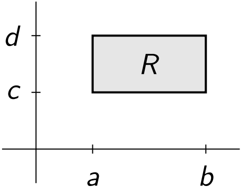
[A coming attraction: Theorem 7.5 in Chapter IV says much more.]

# Some notation of Wilhelm Wirtinger (1865–1945)

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

If $f$ is analytic, then the Cauchy–Riemann equations imply that $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} = f'$ and $\frac{\partial f}{\partial \bar{z}} = 0$.

# The simplest version of Cauchy's integral theorem




For $f$ analytic on a closed rectangle,

$$
\begin{aligned}
0 &= \iint_R i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\,dx\,dy \quad \text{(by the Cauchy–Riemann equations)} \\
&= \int_c^d [f(b,y) - f(a,y)]\; i\,dy + \int_a^b [f(x,c) - f(x,d)]\;dx \\
&= \int_{\partial R} f(z)\,dz
\end{aligned}
$$

[A coming attraction: Theorem 6.6 in Chapter IV says much more.]

# Assignment due next class

Suppose $G$ is an open subset of $\mathbb{C}$, and $f\colon G \to \mathbb{C}$ is an analytic function, expressed as $u(x,y) + iv(x,y)$.

1. Show that the determinant of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

   at a point $z$ in $G$ is equal to $|f'(z)|^2$.

2. Let $G^*$ denote the reflection of $G$ across the real axis: namely, $G^* = \{\, z \in \mathbb{C} : \overline{z} \in G \,\}$. Define a function $f^*\colon G^* \to \mathbb{C}$ by setting $f^*(z)$ equal to $\overline{f(\overline{z})}$. Show that $f^*$ is an analytic function on the set $G^*$.
   [This problem is Exercise 19 in §2 of Chapter III.]