Some standard examples of entire functions

$$\exp(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$
$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$
$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

By Proposition III.2.5, the derivatives are as expected from the corresponding real functions: $\frac{d}{dz}e^z = e^z$ and $\frac{d}{dz}\cos(z) = -\sin(z)$ and $\frac{d}{dz}\sin(z) = \cos(z)$.

Euler's identities

$$
\begin{aligned}
e^{iz} &= 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \cdots \\
&= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + i\left(z - \frac{z^3}{3!} + \cdots\right) \\
&= \cos(z) + i\sin(z).
\end{aligned}
$$

Averaging e^{iz} and $\pm e^{-iz}$ shows that

$$
\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.
$$

These formulas, due to Leonhard Euler (1707–1783), are found (for z real) in his 1748 book, *Introductio in analysin infinitorum*.

More identities

Real-variable identities like

$$e^{x+y} = e^x e^y \qquad \text{and}$$
$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

continue to hold for complex arguments and can be proved in several ways:

- by direct power series manipulations,
- by using the differential equations satisfied by the functions,
- by the "principle of persistence of functional relations" (see the future Corollary IV.3.8).

Remark

The second real identity above is a corollary of the complexified version of the first identity. (Replace x and y by ix and iy and take the imaginary part.)

Warning

Properties of the real exp, sin, and cos functions that involve *inequalities* do *not* carry over to the complex functions.
For example:

- $e^x > 0$ when x is real, but the range of the complex exponential function is $\mathbb{C} \setminus \{0\}$.
- $|\sin(x)| \leq 1$ when x is real, but $|\sin(i)| > 1$, and the range of the complex sine function is all of $\mathbb{C}$.

Inverse functions

Since $\sin(z)$ and $\cos(z)$ are periodic with period 2π, and $\exp(z)$ is periodic with period $2\pi i$, these functions cannot have global inverses. They do have infinitely many local inverses.

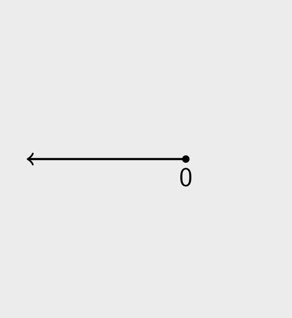
Definition III.2.18 says that a *branch of the logarithm* consists of

1. a choice of a suitable connected open set G, and
2. a continuous function $f\colon G \to \mathbb{C}$ such that $e^{f(z)} = z$ for every z in G.

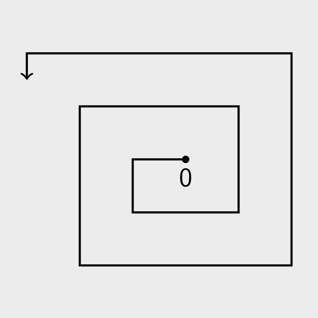
The value of $f(z)$ has to be $\ln|z| + i\arg(z)$, but $\arg(z)$ is determined only up to addition of an integer multiple of 2π.

Since $\arg(z)$ has to be continuous on G, the region G cannot contain a loop that encloses the origin.

Two regions where arg(z) can be defined continuously



$-\pi < \arg(z) < \pi$
(principal value)



$\arg(z)$ continuous
but unbounded

Assignment due next class

- Show that $\sin(z)$ is a bijection from the half-strip where $-\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}$ and $\mathrm{Im}(z) > 0$ to the half-plane where $\mathrm{Im}(w) > 0$.