Interpretation of the derivative of a path

If $\gamma\colon [a,b] \to \mathbb{C}$ is a smooth path, and t is a point at which $\gamma'(t) \neq 0$, then $\gamma'(t)$ represents the direction of the line tangent to the (image of the) path γ.

Now suppose G is an open set containing the image of γ, and $f\colon G \to \mathbb{C}$ is a real-differentiable function. What is the tangent vector to the image curve $f \circ \gamma$?

The real chain rule implies that

$$(f \circ \gamma)'(t) = \frac{\partial f}{\partial z}(\gamma(t))\gamma'(t) + \frac{\partial f}{\partial \bar{z}}(\gamma(t))\overline{\gamma'(t)}.$$

So if f is an analytic function ($\partial f / \partial \bar{z} = 0$), then the tangent vector to the image curve is the product of f' and γ'.

Key deduction: If f is analytic, and if $f' \neq 0$, then the angle at which two curves cross is preserved under mapping by f.

Conformal mapping

If G is open, and $f\colon G \to \mathbb{C}$ is a real-differentiable mapping that preserves the angles at which curves cross (both magnitude and orientation), then f is *conformal*.

Equivalently, f is analytic, and the derivative f' is never equal to 0.

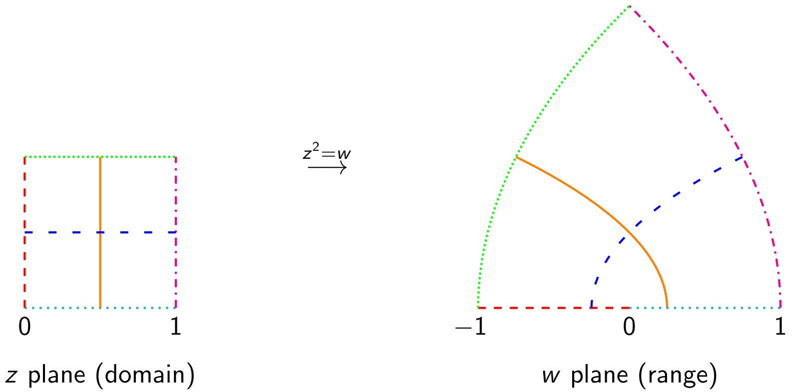
Future Theorem IV.7.4 implies that if f is analytic, then $f'(z_0) \neq 0$ iff f is locally injective in a neighborhood of z_0.

Warning: Some authors require conformal mappings to be *globally* injective.

Some examples

Rotation and translation are rigid motions, hence conformal. Dilation is conformal. The inversion $z \mapsto 1/z$ is conformal on $\mathbb{C} \setminus \{0\}$, since the derivative $-1/z^2$ is never equal to 0.

Conformal mapping example: Geometry of z^2



Conformality fails at the origin: the derivative equals zero there.

Assignment due next class

1. Explain why every Möbius transformation $z \mapsto \dfrac{az+b}{cz+d}$ (where $ad - bc \neq 0$) is conformal on its domain.
2. Show that if f is analytic on an open subset of $\mathbb{C} \setminus \{0\}$, and $z = re^{i\theta}$, then
$$z\,\frac{\partial f}{\partial z} = r\,\frac{\partial f}{\partial r} = -i\,\frac{\partial f}{\partial \theta}$$
(a version of the Cauchy–Riemann equations in polar coordinates).