# A symmetrized version of Rouché's theorem

## Theorem (Theodor Estermann, 1962)

*If $G$ is a simply connected region, and $\gamma$ is a simple closed rectifiable curve in $G$ oriented counterclockwise, and $f$ and $g$ are analytic functions on $G$, and*

$$|f(z) + g(z)| < |f(z)| + |g(z)| \qquad \text{when } z \text{ is on } \gamma,$$

*then $f$ and $g$ have the same number of zeros inside $\gamma$.*

## Remarks

- The hypothesis implies that $f(z)$ and $g(z)$ are nonzero on $\gamma$.
- The hypothesis says that the triangle inequality is *strict*.
- Assuming that $|f(z) - g(z)| < |f(z)| + |g(z)|$ is just as good.
- If $g = f + \varphi$, then the original Rouché theorem follows.
- Theorem V.3.8 is a generalization.

# Homotopy proof of the symmetric Rouché theorem

When $0 \leq t \leq 1$, consider

$$\frac{1}{2\pi i} \int_\gamma \frac{(1-t)f'(z) + t\,g'(z)}{(1-t)f(z) + t\,g(z)}\,dz.$$

The hypothesis that equality does not hold in the triangle inequality implies that the denominator is nonzero on $\gamma$.

The integral is an integer that depends continuously on $t$, hence is constant.

When $t = 0$, the integral represents the number of zeros of $f$ inside $\gamma$. When $t = 1$, the integral represents the number of zeros of $g$ inside $\gamma$.

# Local injectivity

**Theorem**

*If $f$ is analytic in a neighborhood of $z_0$, and $f'(z_0) \neq 0$, then there is a (smaller) neighborhood of $z_0$ on which $f$ is injective.*

**Proof.**

Without loss of generality, suppose $z_0 = 0 = f(z_0)$.
Choose a radius $r$ small enough that $f(z) \neq 0$ when $0 < |z| \leq r$.
When $|b| < \min\{|f(re^{i\theta})| : 0 \leq \theta \leq 2\pi\}$, the expression

$$\frac{1}{2\pi i} \int_{|z|=r} \frac{f'(z)}{f(z) - b} \, dz$$

is continuous in $b$ and integer valued, hence constant.
So in the disk of radius $r$, the function $f$ takes each value $b$ close to 0 the same number of times as $f$ takes the value 0: once.
So $f$ maps a smaller disk bijectively onto a neighborhood of 0. □

# Assignment (not to hand in)

In preparation for the second exam
(which takes place in class on Thursday, November 8),
make a list of the main concepts and theorems covered since the first exam.