Reminder

Exam 2 takes place in class on Thursday, November 8.

Please bring your own paper to the exam.

The mapping z^m

$z \mapsto z^m$ maps a disk centered at 0 to a disk centered at 0.

Each nonzero point b in the image has m distinct preimages.

From last time, we know that each nonzero preimage point has a neighborhood that maps biholomorphically onto a neighborhood of b.

The mapping z^m is the prototype of a “branched covering,” the branch point being 0.

Lemma

If an analytic function f has a zero of order m at 0, then there exists an analytic function g in a neighborhood of 0 such that g has a simple zero at 0, and $f(z) = g(z)^m$.

Proof.

Expand in a power series:

$$f(z) = c_m z^m + c_{m+1} z^{m+1} + \cdots = z^m (c_m + c_{m+1} z + \cdots).$$

The second factor is nonzero near the origin, hence can be written in the form $e^{h(z)}$ for some analytic function h.

Take $g(z)$ to be $z\, e^{h(z)/m}$. $\square$

More on the local behavior of analytic functions (Theorem IV.7.4)

Suppose $f(z)$ has a zero of order m when $z = 0$.

By the lemma, write $f(z)$ in a neighborhood of the origin as $g(z)^m$ for some analytic function g with a simple zero at 0.

From last time, the function g maps some neighborhood of the origin bijectively onto a disk centered at 0.

Therefore f maps some neighborhood of the origin m-to-1 onto a disk centered at 0, and f is locally a branched covering map.

Corollary: open mapping property

Theorem (IV.7.5)

If G is a connected open set, and $f: G \to \mathbb{C}$ is analytic, then the image of f is either

1. *an open subset of $\mathbb{C}$ (the usual case), or*
2. *a single point (the exceptional case, when f is a constant function).*