Announcements

- The comprehensive final examination takes place in this room on the morning of Wednesday, 12 December, 8:00–10:00.
- Please bring paper to the exam.
- Upcoming office hours: 3:00–4:00 on the afternoons of Wednesday 12/5, Monday 12/10, and Tuesday 12/11; also by appointment.

Review and preview: The complex numbers

Review: $\mathbb{C}$ is

- an algebraically closed field
- not an ordered field
- a complete metric space
- a vector space

Preview: Some generalizations of $\mathbb{C}$ are

- Hamiltonian's quaternions $ai + bj + ck + d$, where a, b, c, and d are real numbers, and $-1 = i^2 = j^2 = k^2 = ijk$ (a division algebra that is not commutative)
- Riemann surfaces (one-dimensional complex manifolds)
- the vector space $\mathbb{C}^n$

Review: Characterizations of analytic functions

Subject to suitable additional hypotheses or restrictions, the following properties are more or less equivalent to analyticity.

- Cauchy–Riemann equations
- conformality
- Morera's theorem
- local representation by a convergent power series
- representation by the Cauchy integral formula
- representation by an integral with a free parameter

Preview: Construction of analytic functions

- Riemann mapping theorem: If G is a simply connected domain other than $\mathbb{C}$, then there exists a biholomorphic map (an analytic bijection) from G onto the unit disk.
- Theorem of Weierstrass: The zeros of an analytic function can be prescribed almost arbitrarily.
- Theorem of Mittag-Leffler: Isolated singularities can be prescribed almost arbitrarily.
- Factorization theorems of Weierstrass and Hadamard: an entire function can almost be reconstructed from its zeros.
- Theorem of Runge: On a simply connected domain, an analytic function is a limit of a sequence of polynomials.

Preview: Construction of harmonic functions

A function u is *harmonic* if $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

On simply connected regions, real-valued harmonic functions are the same as real parts of analytic functions.

The Dirichlet problem: Given a continuous function φ on the boundary of a region, find a harmonic function u in the region such that the limit of u at the boundary equals φ.

The Dirichlet problem is solvable when the boundary of the region is sufficiently nice.

Review: The range of a nonconstant analytic function

- open-mapping theorem
- maximum-modulus theorem
- Casorati–Weierstrass theorem

Preview: The range of a nonconstant analytic function

- Picard's little theorem: The range of an entire function is either a point, all of $\mathbb{C}$, or $\mathbb{C}$ minus one point.
- Picard's great theorem: In every neighborhood of an essential singularity, an analytic function takes every value—with one possible exception—infinitely often.
- Bloch's theorem: There exists a positive number b with the following property. For every function f analytic in the unit disk and normalized such that $|f'(0)| = 1$, the range of f contains a schlicht disk of radius b. The supremum of such numbers b is *Bloch's constant*, known to exceed $\sqrt{3}/4 \approx 0.433$ and conjectured since 1936 to equal

$$\sqrt{\frac{\sqrt{3}-1}{2} \cdot \frac{\Gamma(1/3)\,\Gamma(11/12)}{\Gamma(1/4)}} \approx 0.4719.$$