Examination 2

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Give an example of a closed curve γ such that the integrals $\displaystyle\int_\gamma \frac{8}{z-11}\,dz$ and $\displaystyle\int_\gamma \frac{11}{z-8}\,dz$ are well defined, equal, and nonzero.

   Solution. Equality of the integrals implies that $8n(\gamma;11) = 11n(\gamma;8)$. The simplest way to make this equality hold (with both sides nonzero) is to choose γ to have winding number 11 about the point 11 and winding number 8 around the point 8. There are many ways to write down such a curve γ. Here is one:

$$\gamma(t) = \begin{cases} 10 + 3e^{32\pi i t}, & \text{when } 0 \le t \le 1/2, \\ 11 + 2e^{12\pi i (t-\frac{1}{2})}, & \text{when } 1/2 \le t \le 1. \end{cases}$$

   This formula describes a curve that first goes 8 times around a circle with center 10 and radius 3 (both of the points 8 and 11 are inside this circle) and then goes 3 times around a circle with center 11 and radius 2 (only the point 11 is inside this circle).

2. Suppose G is a simply connected open set, and f is an analytic function on G without zeros. You know a theorem stating that there exists a logarithm of f, that is, an analytic function g such that $e^{g(z)} = f(z)$ when $z \in G$.

   (a) If f is injective, must g be injective?

   (b) If g is injective, must f be injective?

   Explain your reasoning.

   Solution.

   (a) Yes. If z_1 and z_2 are points for which $g(z_1) = g(z_2)$, then $e^{g(z_1)} = e^{g(z_2)}$, so $f(z_1) = f(z_2)$, and injectivity of f implies that $z_1 = z_2$. Thus injectivity of f implies injectivity of g.

   (b) No. You saw a counterexample in class. If G is the plane with a cut along the non-positive real axis, and $f(z) = z^2$, then $g(z)$ can be taken to be $2\log(z)$, where $\log(z)$ denotes the principal branch of the logarithm. Indeed, $e^{2\log(z)} = (e^{\log(z)})^2 = z^2$. The function $2\log(z)$ is injective on G, but the function z^2 is not.

   For an even simpler counterexample, take G to be $\mathbb{C}$ and $f(z)$ to be e^z and $g(z)$ to be z. Then g is injective, but f is not injective: indeed, the exponential function is periodic with period $2\pi i$.

3. How many zeros does the function $z^{2018} + 11z^8 + e^z$ have in the annulus where $1 < |z| < 2$? Explain how you know.
   (As usual, zeros are to be counted according to multiplicity.)

Solution. The strategy is to determine the number of zeros inside the disk of radius 2 and then subtract the number of zeros inside the disk of radius 1.

When $|z| = 2$, the power $|z|^{2018}$ is astronomically large, and $|11z^8 + e^z| < 3000$. Therefore the function is a small perturbation of z^{2018}, so Rouché's theorem implies that the number of zeros of the function in the disk of radius 2 matches the number of zeros of z^{2018}, namely, 2018.

When $|z| = 1$, the term $11z^8$ has absolute value equal to 11, and $|z^{2018} + e^z| < 4$. The function is a small perturbation of $11z^8$, so Rouché's theorem implies that there are 8 zeros inside the disk of radius 1.

Subtracting shows that the function has 2010 zeros in the annulus.

4. In some disk with center $11 + 8i$, the function $\dfrac{1}{\cos(z)}$ can be represented by a Taylor series $\sum_{n=0}^{\infty} c_n(z - 11 - 8i)^n$. You know a theorem guaranteeing the existence of a radius R such that this series converges when $|z - 11 - 8i| < R$ and diverges when $|z - 11 - 8i| > R$. Determine the greatest integer less than or equal to R.

Solution. The exact value of R is the radius of the largest disk centered at $11 + 8i$ in which the function is analytic. The function fails to be analytic at the zeros of the cosine function, which lie on the real axis at odd integer multiples of $\pi/2$. Therefore R is the distance from $11 + 8i$ to the closest odd integer multiple of $\pi/2$.

Observe that 11 is a close approximation to $7\pi/2$, because $22/7$ is a close approximation to π. The radius R is exactly equal to $\sqrt{(11 - 7\pi/2)^2 + 8^2}$, and the greatest integer less than or equal to R is 8.

5. Prove there is *no* analytic function f on the disk $\{ z \in \mathbb{C} : |z| < 2018 \}$ such that

$$f(1/n) = \begin{cases} 1/n^8, & \text{when } n \text{ is an even natural number,} \\ 1/n^{11}, & \text{when } n \text{ is an odd natural number.} \end{cases}$$

Solution. Argue by contradiction. Suppose there were such a function f. By continuity, $f(0) = 0$.

Method 1. The function $f(z) - z^8$ has zeros at the reciprocals of the even natural numbers, which accumulate at the origin. Since the zeros of the analytic function $f(z) - z^8$ are not isolated, this function is identically zero: $f(z) \equiv z^8$. By the same reasoning, the second clause of the definition of f implies that $f(z) \equiv z^{11}$. Contradiction.

Method 2. The function has a power series expansion $\sum_{n=0}^{\infty} c_n z^n$. But $c_0 = 0$, because $f(0) = 0$. The definition of f shows that there is a sequence $\{z_n\}_{n=1}^{\infty}$ tending to 0 (namely, the reciprocals of even natural numbers) such that $\lim_{n\to\infty} \dfrac{f(z_n)}{z_n} = 0$. Therefore $c_1 = 0$. Similarly, $0 = c_2 = \cdots = c_7$, and $c_8 = 1$. But now taking the sequence $\{z_n\}_{n=1}^{\infty}$ to be the reciprocals of the odd natural numbers shows that $c_8 = 0$. Contradiction.

6. (a) A student makes an error by claiming that if f is an analytic function on a connected open set G, then $\int_\gamma f(z)\,dz = 0$ for every simple closed smooth curve γ in G. Show that the claim is false by giving a counterexample. (You get to choose G and f and γ.)

   (b) The student makes more error by claiming that if g is a continuous function on a connected open set G, and if there exists a simple closed smooth curve γ in G for which $\int_\gamma g(z)\,dz = 0$, then g is analytic on G. Show that this claim is false too by giving a counterexample.

Solution.

(a) If G is additionally simply connected, then the claim is a correct version of Cauchy's theorem. So to create a counterexample, you need a region G that has a hole and a curve γ that has nonzero winding number about the hole.

For example, take G to be $\mathbb{C} \setminus \{0\}$ and γ to be the unit circle and $f(z)$ to be $1/z$. Then f is analytic in G, and $\int_\gamma f(z)\,dz = 2\pi i \neq 0$.

(b) Morera's theorem says that g will be analytic if $\int_\gamma g(z)\,dz = 0$ for *every* closed curve γ (or for sufficiently many closed curves). The goal of the problem is to exhibit a non-analytic function whose integral around *some* simple closed curve equals 0.

The natural first try for $g(z)$ is the non-analytic function $\overline{z}$, but this attempt fails. Indeed, Green's theorem (from two-dimensional real calculus) implies that $\int_\gamma \overline{z}\,dz$ equals $2i$ times the area of the region inside γ, which cannot be zero.

Here are some examples that do work.

- The function $|z|$ is nowhere analytic on $\mathbb{C}$. Take the curve γ to be the unit circle. On γ, the function $|z|$ matches the constant function 1 (which is analytic!), so $\int_\gamma |z|\,dz = \int_\gamma 1\,dz = 0$.
- Being real-valued, the function $(\operatorname{Re} z)^2$ is nowhere analytic. Again take γ to be the unit circle. Symmetry (or explicit calculation) shows that $\int_\gamma (\operatorname{Re} z)^2\,dz = 0$.

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- If $G = \mathbb{C} \setminus \{0\}$, then the function $1/\overline{z}$ is continuous on G and nowhere analytic. Again take γ to be the unit circle. Since $z\,\overline{z} = 1$ on γ, and z is analytic, $\int_\gamma 1/\overline{z}\,dz = \int_\gamma z\,dz = 0$.
- The preceding examples are nowhere analytic functions. You could alternatively satisfy the demands of the problem by specifying a function that is analytic on part but not all of the region. For instance, suppose $G = \mathbb{C}$, and

$$g(z) = \begin{cases} 0, & \text{when } \operatorname{Re}(z) < 0, \\ \operatorname{Re}(z), & \text{when } \operatorname{Re}(z) \geq 0. \end{cases}$$

  Then g is continuous on G, and g is analytic when $\operatorname{Re}(z) < 0$ but not when $\operatorname{Re}(z) > 0$. If γ is any closed curve contained in the left-hand half-plane, then $\int_\gamma g(z)\,dz = \int_\gamma 0\,dz = 0$.