Final Examination

Instructions: Write solutions to any **six** of the following seven problems. If you attempt all seven problems, please indicate which six you want graded.

1. State

   (a) Cauchy's formula for the radius of convergence of a power series,

   (b) some version of Cauchy's integral formula, and

   (c) some theorem (from this course) *not* named after Cauchy.

   Solution. See the textbook for statements of theorems. Some popular ones for part (c) are the Casorati–Weierstrass theorem, Liouville's theorem, Morera's theorem, Picard's theorems, Rouché's theorem, and the Schwarz lemma.

2. Consider the following complex numbers:

   (a) $(1+i)^{2018}$ (b) $\left(\sqrt{3}+i\right)^{12}$ (c) the principal value of 12^i.

   Which of these three complex numbers has the largest real part? Explain how you know.

   Solution. Writing $1+i$ as $\sqrt{2}\,e^{\pi i/4}$ shows that $(1+i)^{2018} = 2^{1009}e^{\pi i/2} = 2^{1009}i$, so $\operatorname{Re}(1+i)^{2018} = 0$. Similarly writing $\sqrt{3}+i$ as $2e^{\pi i/6}$ shows that $\left(\sqrt{3}+i\right)^{12} = 2^{12}$. Finally, the principal value of 12^i is $e^{i\ln 12}$, a complex number of absolute value equal to 1, hence a number whose real part does not exceed 1. Therefore the complex number in part (b) has the largest real part of the three.

   Remark. The real part of the principal value of 12^i is actually negative!

3. Suppose $\gamma\colon [0,1] \to \mathbb{C}$ is the unit circle, that is, $\gamma(t) = e^{2\pi i t}$. Give an example of an analytic function f on $\mathbb{C}\setminus\{0\}$ such that

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = 12 \qquad \text{and} \qquad \frac{1}{2\pi i}\int_\gamma (f(z))^2\,dz = 2018.$$

   Solution. The answer is not unique. The simplest such function has the form $\dfrac{a}{z}+b$ for suitable constants a and b. Taking a equal to 12 gives the correct value for the first integral. Since $\left(\dfrac{a}{z}+b\right)^2 = \dfrac{a^2}{z^2}+b^2+\dfrac{2ab}{z}$, setting $2ab$ equal to 2018 gives the correct value for the second integral. Therefore $b = \dfrac{1009}{12}$, and $f(z) = \dfrac{12}{z}+\dfrac{1009}{12}$.

   Remark. Two other popular answers are $\dfrac{12}{z}\exp\left(\dfrac{1009z}{144}\right)$ and $\dfrac{12}{z}+1009z+\dfrac{1}{z^2}$.

Final Examination

4. Determine the value of $\sup\left\{ \left|\dfrac{\sin(z)}{z}\right| : \quad z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1 \right\}$.

Solution. The singularity of $\dfrac{\sin(z)}{z}$ at the origin is removable: if the value at 0 is defined to be 1 (the value of the limit), then the function becomes entire. The maximum-modulus theorem implies that the required supremum is attained on the boundary of the unit disk, where $|z| = 1$. The value is therefore identical to $\max\{\, |\sin(z)| : |z| = 1 \,\}$. Here are two ways to prove that this maximum is attained when $z = i$.

Method 1. Use the Maclaurin series of $\sin(z)$ and the triangle inequality to observe that

$$\begin{aligned} |\sin(z)| &= \left| z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right| \\ &\le |z| + \frac{|z|^3}{3!} + \frac{|z|^5}{5!} + \cdots \\ &= 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots \qquad \text{when } |z| = 1 \\ &= \sinh(1). \end{aligned}$$

This upper bound on $|\sin(z)|$ is attained when $z = \pm i$, so $\sinh(1)$ is the required value.

Method 2. Since $|\sin(-z)| = |\sin(z)|$, it suffices to consider those values of z having nonnegative real part. Since $|\sin(\overline{z}\,)| = \left|\overline{\sin(z)}\right| = |\sin(z)|$, it suffices to consider values of z with nonnegative imaginary part. The problem thus reduces to determining $\max\{\, |\sin(e^{i\theta})| : 0 \le \theta \le \pi/2 \,\}$, which is a problem of one-variable real calculus.

The claim now is that the function $|\sin(e^{i\theta})|$ is strictly increasing on the interval where $0 < \theta < \pi/2$, so the maximum is attained when $\theta = \pi/2$, and the maximum value equals $|\sin(i)|$, equivalently $(e^1 - e^{-1})/2$, or $\sinh(1)$. To verify the claim, it suffices to show that the derivative of $|\sin(e^{i\theta})|^2$ is positive when $0 < \theta < \pi/2$.

Write $e^{i\theta}$ as $x + iy$, where $x = \cos(\theta)$ and $y = \sin(\theta)$. Notice that $0 < x < 1$ and $0 < y < 1$ when $0 < \theta < \pi/2$. Since

$$|\sin(x + iy)|^2 = (\sin(x))^2 + (\sinh(y))^2,$$

the derivative with respect to θ equals

$$2\sin(x)\cos(x)\,\frac{dx}{d\theta} + 2\sinh(y)\cosh(y)\,\frac{dy}{d\theta},$$

or

$$-2y\sin(x)\cos(x) + 2x\sinh(y)\cosh(y).$$

Compare the (positive) factors in the two terms: $0 < y < \sinh(y)$, and $0 < \sin(x) < x$ since $0 < x < 1 < \pi$, and $0 < \cos(x) < 1 < \cosh(y)$ since $0 < x < 1 < \pi/2$ and $y \ne 0$; so the expression for the derivative is indeed positive.

Final Examination

5. According to the "method of partial fractions," there exist constants A_1, A_2, …, A_{2018} such that

$$\frac{z^{12}}{(z-1)(z-2)\cdots(z-2018)} = \frac{A_1}{z-1} + \frac{A_2}{z-2} + \cdots + \frac{A_{2018}}{z-2018}.$$

The coefficient A_n is simply the residue of the left-hand side at the simple pole n. Prove that $A_1 + A_2 + \cdots + A_{2018} = 0$.

Solution. Method 1. Integrate over a circle centered at the origin of radius R greater than 2018, and divide by $2\pi i$. What appears on the right-hand side is precisely the sum $A_1 + A_2 + \cdots + A_{2018}$. What appears on the left-hand side has absolute value less than

$$\frac{R^{13}}{(R-1)(R-2)\cdots(R-2018)}.$$

This expression tends to 0 when $R \to \infty$.

Method 2. The problem can be solved by polynomial algebra without invoking any complex analysis. Multiply both sides by the denominator $(z-1)(z-2)\cdots(z-2018)$ and observe that $A_1 + A_2 + \cdots + A_{2018}$ equals the coefficient of z^{2017} on the right-hand side. But the coefficient of z^{2017} on the left-hand side is 0.

6. Suppose f is an entire function having the property that $f(z^{12}) = (f(z))^{2018}$ for every z. Prove that f must be a constant function.

Solution. Method 1. Observe that $f(0) = (f(0))^{2018}$, so either $f(0) = 0$ or $f(0)$ is a 2017th root of unity. Seeking a contradiction, suppose that $f(z)$ is not constantly equal to $f(0)$.

If $f(0) = 0$, then f has a zero at the origin of order n for some positive integer n. But then $f(z^{12})$ has a zero of order $12n$, and $(f(z))^{2018}$ has a zero of order $2018n$. So $12n = 2018n$: contradiction.

If $f(0) \neq 0$, let n be the smallest positive integer for which $c_n \neq 0$, where c_n is the coefficient of order n in the Maclaurin series of f. In other words, $f(z)$ has a convergent power series expansion $f(0) + c_n z^n + \cdots$. Then

$$f(z^{12}) = f(0) + c_n z^{12n} + \cdots, \qquad \text{and}$$
$$(f(z))^{2018} = (f(0))^{2018} + 2018(f(0))^{2017} c_n z^n + \cdots.$$

The second power series has a z^n term, but the first one does not: contradiction.

Method 2. Taking the derivative shows that $12z^{11} f'(z^{12}) = 2018(f(z))^{2017} f'(z)$. Since the left-hand side equals 0 when $z = 0$, the right-hand side does too, so either $f(0) = 0$ or $f'(0) = 0$.

If $f(0) = 0$, but $f(z)$ is not identically equal to 0, then f has a zero at the origin of some positive order n, and f' has a zero at the origin of order $n-1$. Therefore $z^{11}f'(z^{12})$ has a zero of order $11 + 12(n-1)$, but $(f(z))^{2017}f'(z)$ has a zero of order $2017n + n - 1$: contradiction.

If $f(0) \neq 0$, but $f'(0) = 0$, and $f'(z)$ is not identically equal to 0 (that is, f is not constant), then f' has a zero at the origin of some positive order n. So $z^{11}f'(z^{12})$ has a zero of order $11 + 12n$, but $(f(z))^{2017}f'(z)$ has a zero of order n: contradiction.

Method 3. Let M denote $\max\{\, |f(z)| : |z| \le 1 \,\}$. Suppose this maximum is attained at the point w in the closed unit disk. Then w^{12} is still a point in the closed unit disk, so

$$M^{2018} = |f(w)|^{2018} = |(f(w))^{2018}| = |f(w^{12})| \le M.$$

Therefore $M \le 1$, that is, $|f(z)| \le 1$ when $|z| \le 1$.

If f has no zeros in the closed unit disk, then the same argument applies to $1/f$, whence $1/|f(z)| \le 1$ when $|z| \le 1$. Combining this inequality with the previous one shows that $|f(z)| = 1$ when $|z| \le 1$, so f maps the open unit disk into a set with empty interior. The open-mapping theorem now implies that f is a constant function.

The remaining possibility is that there exists a point z_0 in the closed unit disk such that $f(z_0) = 0$. If $z_0 = 0$, then f must be identically equal to 0, else the order n of the zero of f would have the impossible property that $12n = 2018n$ (as in Method 1). Suppose, then, that $z_0 = re^{i\theta}$, where $0 < r \le 1$ and $0 < \theta \le 2\pi$. Define z_1 to be $r^{1/12}e^{i\theta/12}$. The defining equation of f implies that $0 = f(z_0) = f(z_1^{12}) = (f(z_1))^{2018}$, so $f(z_1) = 0$. In general, define z_n to be $r^{1/12^n}e^{i\theta/12^n}$. Since $z_n^{12} = z_{n-1}$, induction implies that $f(z_n) = 0$ for every positive integer n. The sequence $\{z_n\}_{n=1}^{\infty}$ consists of points different from 1 (the sneaky choice of θ ensures that if $z_0 = 1$, then $z_1 = e^{i\pi/6} \neq 1$) but converging to 1, so the point 1 is a nonisolated zero of f. Consequently, the function f must be identically equal to 0.

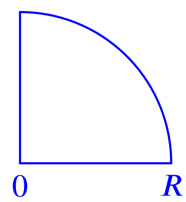
7. A student analyzes the real integral $\int_0^\infty e^{-x^4}\,dx$ as follows. "Make the change of variable $x = it$. Then $x^4 = t^4$ and $dx = i\,dt$. When x goes from 0 to ∞, so does t, whence

$$\int_0^\infty e^{-x^4}\,dx = i\int_0^\infty e^{-t^4}\,dt.$$

The name of the dummy integration variable does not matter, so $\int_0^\infty e^{-x^4}\,dx$ equals i times itself, hence equals 0. But WolframAlpha says that $\int_0^\infty e^{-x^4}\,dx = \Gamma(5/4) \approx 0.906$. Did I just show that mathematics is inconsistent?" Restore the student's sanity for the holidays by explaining the flaw in the reasoning.

Solution. To analyze the problem carefully, treat the improper integral as a limit and keep track of the integration path in the complex plane. The original integral can be viewed as $\lim_{R\to\infty} \int_{\gamma_R} e^{-z^4}\,dz$, where $\gamma_R \colon [0, R] \to \mathbb{C}$ is the identity path, namely, $\gamma_R(s) = s$. Setting z equal to iw converts the problem into $\lim_{R\to\infty} i \int_{\tilde{\gamma}_R} e^{-w^4}\,dw$, where $\tilde{\gamma}_R \colon [0, R] \to \mathbb{C}$ is a different path, namely, $\tilde{\gamma}_R(s) = -is$. Since the paths are different, the integrals $\int_{\gamma_R} e^{-z^4}\,dz$ and $\int_{\tilde{\gamma}_R} e^{-w^4}\,dw$ are (possibly) different. The flaw in the student's reasoning is to overlook the distinction between γ_R and $\tilde{\gamma}_R$.

More can be said by considering the path shown in the figure (a quarter circle).



The integral of e^{-z^4} over this closed path equals 0, since the integrand is analytic inside the path. Therefore $\int_0^R e^{-x^4}\,dx - i\int_0^R e^{-t^4}\,dt$ equals the negative of the integral of e^{-z^4} over the curved part of the path. The student's mistake amounts to claiming that the integral over the curved part of the path has limit equal to 0 when $R \to \infty$. Since z^4 maps the quarter circle to a full circle, the function e^{-z^4} has very large absolute value on part of the integration path, so there is no reason to expect the limit of this integral to be 0 (and in fact the limit is not 0).

Remark. Integrating by parts shows that

$$\int_0^\infty e^{-x^4}\,dx = \int_0^\infty 4x^4 e^{-x^4}\,dx,$$

and now substituting u for x^4 shows that

$$\int_0^\infty e^{-x^4}\,dx = \int_0^\infty u^{1/4} e^{-u}\,du.$$

Since $\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}\,du$ when $\mathrm{Re}(z) > 0$, this calculation confirms WolframAlpha's claim that $\int_0^\infty e^{-x^4}\,dx = \Gamma(5/4)$.