Theory of Functions of a Complex Variable I

Instructions Solve **six** of the following seven problems. Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "find" or that asks a question requires an explanation to support the answer. Please explain your reasoning in complete sentences.

1. Find the first two nonzero terms of the Laurent series of $\dfrac{1}{z^2(e^z - e^{-z})}$ that is valid in the punctured disk where $0 < |z| < \pi$.

   Solution. This problem is number 5 on page 5 in Section 4.1.

   Since

$$\begin{aligned} e^z - e^{-z} &= (1 + z + \tfrac{1}{2!}z^2 + \tfrac{1}{3!}z^3 + \cdots) - (1 - z + \tfrac{1}{2!}z^2 - \tfrac{1}{3!}z^3 + \cdots) \\ &= 2(z + \tfrac{1}{3!}z^3 + \cdots), \end{aligned}$$

   it follows that

$$\begin{aligned} \frac{1}{z^2(e^z - e^{-z})} &= \frac{1}{2z^3} \cdot \frac{1}{1 + \frac{1}{6}z^2 + \cdots} = \frac{1}{2z^3} \cdot (1 - \tfrac{1}{6}z^2 + \cdots) \\ &= \frac{1}{2z^3} - \frac{1}{12z} + \cdots . \end{aligned}$$

   A different method is shown in the textbook.

2. Classify each isolated singularity of the function $\dfrac{1}{z^2(z+1)} + \sin\left(\dfrac{1}{z}\right)$: is the singularity removable? essential? a pole?

   Solution. This problem is number 10(b) on page 6 in Section 4.1.

   There is an essential singularity at 0 and a simple pole at -1. (Also there is a removable singularity at ∞.)

3. Use the residue theorem to prove that

$$\frac{1}{2\pi} \int_0^{2\pi} (\sin\theta)^{2n}\, d\theta = \frac{(2n)!}{(n!\, 2^n)^2}$$

   when n is a natural number.

   Solution. This problem is number 4(g) on page 11 in Section 4.2.

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Since $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, setting $e^{i\theta}$ equal to z reformulates the integral as a path integral over the unit circle:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} (\sin\theta)^{2n}\,d\theta &= \frac{1}{2\pi}\int_{|z|=1} \left(\tfrac{1}{2i}\right)^{2n} (z - z^{-1})^{2n}\,\frac{dz}{iz} \\ &= \frac{(-1)^n}{2^{2n}} \cdot \frac{1}{2\pi i}\int_{|z|=1} (z - z^{-1})^{2n}\tfrac{1}{z}\,dz.\end{aligned}$$

The only singularity of the integrand is at the origin, and the residue (the coefficient of $\frac{1}{z}$ in the Laurent series) is the constant term in the binomial expansion of $(z - z^{-1})^{2n}$, namely $\binom{2n}{n} z^n(-z^{-1})^n$, or $\binom{2n}{n}(-1)^n$. By the residue theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} (\sin\theta)^{2n}\,d\theta = \frac{(-1)^n}{2^{2n}} \cdot \binom{2n}{n}(-1)^n = \frac{(2n)!}{(n!\,2^n)^2}.$$

4. Suppose that f is analytic in an open neighborhood of the closed unit disk $\overline{D}(0,1)$, and $|f(z)| < 1$ when $|z| \le 1$. Brouwer's fixed-point theorem from topology implies that f has at least one fixed point in the closed unit disk (that is, there exists a point z_0 such that $f(z_0) = z_0$). Use Rouché's theorem to show that in this special setting, the function f has *exactly one* fixed point in the closed unit disk.

   Solution. This problem is part of number 11 on page 13 in Section 4.2. (I have added the remark about Brouwer's theorem.)

   Apply Rouché's theorem to the pair of functions $f(z) - z$ and z and the closed curve $C(0,1)$. Notice first that $f(z) - z$ has no zero on $C(0,1)$, for the triangle inequality implies that $|f(z) - z| \ge |z| - |f(z)| = 1 - |f(z)| > 0$ when $|z| = 1$. Now

$$|(f(z) - z) + z| = |f(z)| < 1 = |z| < |f(z) - z| + |z| \qquad \text{when } |z| = 1,$$

   so the hypothesis of Rouché's theorem is met. Therefore $f(z) - z$ and z have the same number of zeroes inside the unit disk: namely, one. In other words, there is a unique value of z for which $f(z) - z = 0$, that is, for which $f(z) = z$.

5. Suppose that f and g are analytic in an open neighborhood of the closed unit disk $\overline{D}(0,1)$, and f has no zeroes on the unit circle $C(0,1)$. Let the distinct zeroes of f in $D(0,1)$ be $a_1, \ldots, a_n$, and suppose that each of these zeroes is simple (that is, first order). Prove that

$$\frac{1}{2\pi i}\int_{C(0,1)} \frac{f'(z)}{f(z)} g(z)\,dz = \sum_{j=1}^{n} g(a_j).$$

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Solution. This problem is a special case of number 4 on page 17 in Section 4.3.

The residue theorem implies that the left-hand side equals the sum of the residues of the function

$$\frac{f'(z)}{f(z)}g(z) \tag{†}$$

at the zeroes of f inside the unit disk. At a simple zero a_j, the Taylor series of f begins $f'(a_j)(z-a_j)+\cdots$, and the Laurent series of $f'(z)/f(z)$ begins $\frac{1}{z-a_j}+\cdots$. Therefore the Laurent series of $f'(z)g(z)/f(z)$ begins $\frac{g(a_j)}{z-a_j}+\cdots$, and the residue equals $g(a_j)$. Thus the right-hand side of the formula equals the sum of the residues of the function (†) too.

6. Find a linear fractional transformation (a Möbius transformation) that fixes the points 1 and -1 and maps i to 0.

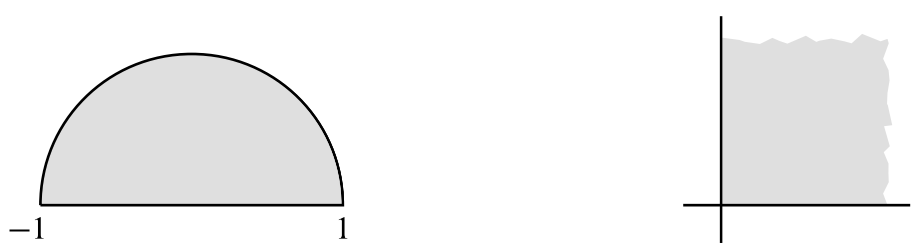
Solution. This problem is number 3(a) on page 20 in Section 4.5.

If the transformation has the form $\frac{az+b}{cz+d}$, then $a+b=c+d$ since the point 1 is fixed, and $-a+b=c-d$ since the point -1 is fixed. Adding these equations shows that $b=c$, and subtracting the equations shows that $a=d$. Moreover, $b=-ai$ since i maps to 0. Therefore the transformation has the following form:

$$\frac{az+b}{cz+d}=\frac{az-ai}{-aiz+a}=\frac{z-i}{-iz+1}=\frac{iz+1}{z+i}.$$

The transformation is unique, but there is more than one way to express the formula, since the value of a can be chosen to be an arbitrary nonzero complex number.

7. Does there exist a linear fractional transformation (a Möbius transformation) that maps the open half-disk $\{\, z\in\mathbb{C} : |z|<1 \text{ and } \operatorname{Im} z>0 \,\}$ onto the open first quadrant? (See the figure below.) Explain.



Solution. Although this problem is not an explicit exercise in the textbook, it is closely related to problem 2 on page 20 in Section 4.5.

There is indeed such a Möbius transformation: the transformation $\dfrac{1+z}{1-z}$ is one example (not unique).

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To see that this transformation does the job, observe first that the coefficients are real, so the extended real axis maps to itself. The point 1 maps to the point at infinity, so the unit circle maps to some line; since Möbius transformations are conformal, this line is perpendicular to the real axis; and since the point -1 maps to 0, the line is the imaginary axis. Since i is a fixed point of the transformation, the upper half of the unit circle maps to the upper half of the imaginary axis. Similar reasoning shows that the upper half-circle with diameter $[b, 1]$, where $-1 < b < 1$, maps to a vertical half-line with x-intercept $\dfrac{1+b}{1-b}$. These half-lines fill up the first quadrant as b varies from -1 to 1. Thus the transformation maps the open upper half-disk onto the open first quadrant.

Alternatively, you could compute that

$$\frac{1+z}{1-z} = \frac{1+z}{1-z} \cdot \frac{1-\overline{z}}{1-\overline{z}} = \frac{1-|z|^2 + 2i\,\operatorname{Im} z}{|1-z|^2}.$$

This transformation maps a point z to a point of the open first quadrant if and only if the image point has both positive real part and positive imaginary part; that is, if and only if both $1-|z|^2 > 0$ and $\operatorname{Im} z > 0$; that is, if and only if z lies in the open upper half-disk. Since Möbius transformations are bijections of the extended complex plane, it follows that the open upper half-disk maps precisely onto to the open first quadrant.

Remark Notice that the boundary of the left-hand diagram has two right angles, while the boundary of the right-hand diagram has only one right angle. The missing right angle in the second diagram is at infinity, where the two coordinate axes meet orthogonally.