

Theory of Functions of a Complex Variable I

Instructions Please solve the following six problems (all of which are exercises excerpted from the textbook). Treat these problems as essay questions: supporting explanation is required.

1. Find every complex number z satisfying the property that $|z - i| = 2z + i$.

Solution. This problem is Exercise 1.21(a) on page 11, solved on page 235. The answer is that $z = \frac{1}{2}(\sqrt{3} - i)$.

2. Let ω denote $e^{2\pi i/3}$, a cube root of 1. Let $g(z)$ denote the product

$$\cos(z) \cos(\omega z) \cos(\omega^2 z).$$

Show that when n is not a multiple of 3, the n th coefficient in the Maclaurin series of g is equal to zero. In other words, there exists an entire function f such that $g(z) = f(z^3)$ for every z .

Solution. This problem is Exercise 4.22 on page 30, solved on page 247. Here is a solution slightly different from the one in the textbook.

Evidently $g(\omega z) = g(z)$ for every z , since $\omega^3 = 1$. Accordingly, if the Maclaurin series for $g(z)$ is $\sum_{n=0}^{\infty} a_n z^n$, then

$$\sum_{n=0}^{\infty} a_n \omega^n z^n = \sum_{n=0}^{\infty} a_n z^n \quad \text{for every } z.$$

Two power series agree identically if and only if the coefficients match, so $a_n \omega^n = a_n$ for every n . Consequently, if $a_n \neq 0$, then $\omega^n = 1$, and this equality holds if and only if n is a multiple of 3. Thus the series has the form $\sum_{k=0}^{\infty} a_{3k} (z^3)^k$.

3. Let C be the circle with center 1 and radius 1 (oriented in the usual counterclockwise direction and traversed once). Evaluate the line integral

$$\int_C \frac{1+z}{1-z} dz.$$

Solution. This problem is Exercise 5.5 on page 33, solved on page 248. The solution in the book suggests parametrizing the integral, for Cauchy's integral formula is not yet known in Section 5. But you know more now, so you can apply either the integral formula or the residue theorem to obtain the answer $-4\pi i$ with almost no work. (The minus sign appears because the denominator of the integrand is $1 - z$ instead of $z - 1$.)

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4. Suppose f is an entire function such that

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta$$

is a bounded function of the radius r . Show that f must be a constant function. (This problem is a variation on Liouville's theorem.)

Solution. This problem is Exercise 7.8 on page 50, solved on page 250.

5. Use the residue theorem to prove that

$$\int_0^{\infty} \frac{1}{x^4 + 6x^2 + 8} dx = \frac{\pi(\sqrt{2} - 1)}{8}.$$

Solution. This problem is Exercise 9.2(c) on page 73, solved on page 256. The integral actually can be evaluated by using techniques from first-year calculus, but using the residue theorem is easier.

6. Determine the number of zeroes of the function $2iz^2 + \sin(z)$ in the rectangle where $|\operatorname{Re}(z)| \leq \pi/2$ and $|\operatorname{Im} z| \leq 1$.
(You may find it useful to know that $\cosh(1) < 1.6$.)

Solution. This application of Rouché's theorem is Exercise 12.3 on page 98, solved on pages 268–269.

Optional bonus problem for extra credit

Let D denote $\mathbb{C} \setminus [-1, 1]$, the complex plane with the closed segment $[-1, 1]$ of the real axis removed. Is it possible to define a holomorphic branch of $\sqrt{z^2 - 1}$ on D ? In other words, does there exist a holomorphic function f on D with the property that $(f(z))^2 = z^2 - 1$ for every z in D ? Explain why or why not.

Solution. There does exist such a function, but constructing it requires some care.

The first try—which fails—is to say that $\sqrt{z^2 - 1} = \exp\{\frac{1}{2} \log(z^2 - 1)\}$, so it suffices to construct $\log(z^2 - 1)$ on D . But there cannot exist a global holomorphic branch of $\log(z^2 - 1)$ on D . Indeed, if there were such a global holomorphic function, then its derivative would be $2z/(z^2 - 1)$, and the integral of this derivative around a closed path in D would be equal to 0. Since

$$\int_{|z|=2} \frac{2z}{z^2 - 1} dz = 4\pi i \neq 0,$$

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the global existence of $\log(z^2 - 1)$ on D is ruled out. The subtlety in the problem is that a global holomorphic branch of $\sqrt{z^2 - 1}$ can be constructed in D even though a global holomorphic branch of $\log(z^2 - 1)$ does not exist.

An intuitive way to see why the required function should exist is to think of $\sqrt{z^2 - 1}$ as $\sqrt{z - 1} \sqrt{z + 1}$. Now $\sqrt{z - 1}$ can be defined on $\mathbb{C} \setminus (-\infty, 1]$, and $\sqrt{z + 1}$ can be defined on $\mathbb{C} \setminus (-\infty, -1]$. Both of these functions will be discontinuous across the segment $(-\infty, -1)$. In each case, the discontinuity as z crosses this segment is merely a change in sign (since the square root of a complex number has only two possible values, which are negatives of each other). Consequently, when the functions $\sqrt{z - 1}$ and $\sqrt{z + 1}$ are multiplied, the discontinuity across the segment $(-\infty, -1)$ evaporates, for the two sign changes cancel each other out.

To construct the function more formally, let $\text{Sqrt}(z)$ denote the principal square root function on $\mathbb{C} \setminus (-\infty, 0]$: namely, $\text{Sqrt}(z) = \exp\{\frac{1}{2} \text{Log}(z)\}$, where Log denotes the principal branch of the logarithm function (determined by taking $\arg(z)$ between $-\pi$ and π). When z lies in $\mathbb{C} \setminus (-\infty, 1]$, the points $z - 1$ and $z + 1$ lie in $\mathbb{C} \setminus (-\infty, 0]$, so the product $\text{Sqrt}(z - 1) \text{Sqrt}(z + 1)$ is a well-defined holomorphic function on $\mathbb{C} \setminus (-\infty, 1]$ whose square equals $z^2 - 1$. On the other hand, $-\text{Sqrt}(z^2 - 1)$ is a well-defined holomorphic function on the intersection of D with the open left-hand half-plane, for when z lies in this intersection, the point $z^2 - 1$ lies in $\mathbb{C} \setminus (-\infty, 0]$. Moreover, the square of $-\text{Sqrt}(z^2 - 1)$ evidently equals $z^2 - 1$. The claim now is that the following definition solves the problem:

$$f(z) = \begin{cases} \text{Sqrt}(z - 1) \text{Sqrt}(z + 1) & \text{when } z \in \mathbb{C} \setminus (-\infty, 1], \\ -\text{Sqrt}(z^2 - 1) & \text{when } z \in D \cap \{z \in \mathbb{C} : \text{Re}(z) < 0\}. \end{cases}$$

What needs to be checked is that the two clauses of the definition agree on the overlap of the two indicated domains: namely, on the open second quadrant and on the open third quadrant. When z lies in the open second quadrant, the point $z^2 - 1$ lies in the open lower half-plane, so $\text{Sqrt}(z^2 - 1)$ lies in the open fourth quadrant, and $-\text{Sqrt}(z^2 - 1)$ lies in the open second quadrant. If z lies in the open second quadrant, and additionally $\text{Re}(z) < -1$, then both the points $z - 1$ and $z + 1$ lie in the open second quadrant, so both $\text{Sqrt}(z - 1)$ and $\text{Sqrt}(z + 1)$ have angles between $\pi/4$ and $\pi/2$, whence the product $\text{Sqrt}(z - 1) \text{Sqrt}(z + 1)$ lies in the second quadrant. For such values of z , the quantities $\text{Sqrt}(z - 1) \text{Sqrt}(z + 1)$ and $-\text{Sqrt}(z^2 - 1)$ agree, for they lie in the same quadrant and have the same square. The coincidence principle then implies that $\text{Sqrt}(z - 1) \text{Sqrt}(z + 1) = -\text{Sqrt}(z^2 - 1)$ for every z in the open second quadrant. Completely analogous reasoning shows that $\text{Sqrt}(z - 1) \text{Sqrt}(z + 1) = -\text{Sqrt}(z^2 - 1)$ when z lies in the open third quadrant.¹ Thus the formula for f is well defined: the two cases agree at all points in the intersection of the two domains.

¹Applying the coincidence principle requires making separate arguments for the second quadrant and for the third quadrant because the open set $\{z \in \mathbb{C} : \text{Re}(z) < 0\} \setminus (-\infty, 1]$ is disconnected.