Exercise on the Weierstrass $\wp$ function

This exercise provides a direct solution (without using series expansions) of the homework problem stating that

$$\det \begin{vmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(z_1+z_2) & -\wp'(z_1+z_2) & 1 \end{vmatrix} = 0, \qquad (*)$$

where $\wp$ is the Weierstrass doubly periodic meromorphic function defined with respect to an arbitrary lattice, and all the terms in the determinant are assumed to be finite (that is, the points z_1, z_2, and $z_1 + z_2$ are not points of the lattice).

1. Deduce from the residue theorem that if f is meromorphic in a simply connected region, if γ is a simple closed curve in the region, if the zeroes of f inside γ are $a_1, \ldots, a_m$ (repeated according to multiplicity), if the poles of f inside γ are $b_1, \ldots, b_n$ (repeated according to multiplicity), and if f has no zeroes or poles on the curve γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)}\,dz = \sum_{j=1}^m a_j - \sum_{k=1}^n b_k.$$

(Compare Exercise 2 on page 174, which you did last semester.)

2. Suppose f is a doubly periodic meromorphic function that has a multiple pole at the origin and no other poles within a fundamental parallelogram centered at the origin. Deduce from part 1 that if the zeroes of f in the parallelogram are $a_1, \ldots, a_m$, then $a_1 + \cdots + a_m = 0$.[1]

3. Fix a lattice and a corresponding Weierstrass $\wp$ function. Let z_1 and z_2 be distinct complex numbers such that z_1, z_2, and $z_1 + z_2$ are not points of the lattice, and consider the determinant function D defined by

$$D(z) = \det \begin{vmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(z) & \wp'(z) & 1 \end{vmatrix}.$$

Show that D is a doubly periodic meromorphic function of z of order 3 that has zeroes at z_1 and z_2. Use part 2 to locate a third zero of D, and deduce equation $(*)$ from the symmetry properties of $\wp$ and $\wp'$.

[1] Correction added 19 April: the conclusion should be that $a_1 + \cdots + a_m \equiv 0$ modulo the lattice; that is, $a_1 + \cdots + a_m$ is an element of the lattice.