Exercise on univalent functions

A holomorphic function that takes no value more than once is called variously one-to-one or univalent or *schlicht*. The latter term is a German word that has no exact English equivalent. (There is a story that a student once asked W. F. Osgood if there were an English term for schlicht functions. Osgood is supposed to have replied, "You *could* call them univalent functions, and everyone would know that you meant schlicht.")

Local univalence

1. Show that if f is a holomorphic function, and a is a point at which the derivative $f'(a) \neq 0$, then there is a neighborhood of a in which f is one-to-one.[1]

2. Give an example of an entire function that is locally one-to-one but not globally one-to-one.[2]

Global univalence

One needs either a formula or some global information to show that a holomorphic function is univalent in the large. Let's consider functions whose domain is the unit disc.

3. Show that the *Koebe function* $z \mapsto \dfrac{z}{(1-z)^2}$ maps the (open) unit disc one-to-one onto the plane minus a slit from $-1/4$ to $-\infty$ along the negative real axis.[3]

4. Suppose that f is holomorphic on a neighborhood of the closure of the unit disc, and f is one-to-one on the *boundary* of the disc. Show that f is one-to-one on the closed disc, and the image of the open disc is the region inside the image of the boundary.[4]

[1] Hint: Use the argument principle. See also Theorem 5.2.2.

[2] Hint: What function do you know whose derivative is nowhere zero?

[3] Hint: $\dfrac{z}{(1-z)^2} = \dfrac{1}{4}\left(\left(\dfrac{1+z}{1-z}\right)^2 - 1\right)$

[4] This is problem 17 on page 178. Hint: Use the hint for the first problem.