

Exercise on the definition of infinite products

You know that the notation $\sum_{n=1}^{\infty} c_n$ means (by definition) $\lim_{N \rightarrow \infty} \sum_{n=1}^N c_n$. The analogous definition for infinite products is more subtle, as the following examples show.

1. Evaluate (a) $\lim_{N \rightarrow \infty} \prod_{n=1}^N (1 - 2^{n-1})$ and (b) $\lim_{N \rightarrow \infty} \prod_{n=1}^N (1 - (n+1)^{-1})$.

A reasonable definition of infinite products should satisfy the fundamental properties that (i) deleting one factor (or finitely many) does not affect whether or not the product converges, and (ii) a product is equal to zero if and only if one of the factors is zero.

2. What would go wrong in 1(a) and 1(b) if we were naively to define

$$\prod_{n=1}^{\infty} c_n \text{ to be equal to } \lim_{N \rightarrow \infty} \prod_{n=1}^N c_n?$$

The correct definition is that $\prod_{n=1}^{\infty} c_n$ converges and equals $\lim_{N \rightarrow \infty} \prod_{n=1}^N c_n$ if that limit exists and if nothing goes wrong: namely, (i) at most finitely many factors c_j are equal to 0, and (ii) the limit is not equal to 0.

3. You know that a necessary (but not sufficient) condition for an infinite series to converge is that the terms of the series tend to zero. What is the analogous necessary (but not sufficient) condition for an infinite product to converge?
4. Since the logarithm of a product equals the sum of the logarithms, one would like to say that $\prod_{n=1}^{\infty} c_n$ converges if and only if $\sum_{n=1}^{\infty} \log c_n$ converges. Can you make this equivalence both precise and correct? (One point to worry about is choosing a branch of the logarithm consistently.)