Exercise on the $\overline{\partial}$-equation

The goal of this exercise is to solve the inhomogeneous Cauchy-Riemann equation (also known as the $\overline{\partial}$-equation) on planar domains.

The claim is that if a function g has a continuous derivative on the closure of a bounded domain Ω in the complex plane, and if

$$f(z) := -\frac{1}{\pi}\iint_\Omega \frac{g(\zeta)}{\zeta - z}\, d\xi\, d\eta, \qquad z \in \Omega, \quad \zeta = \xi + i\eta, \tag{1}$$

then $\partial f/\partial\overline{z} = g$ in Ω.

1. First consider the special case that g has *compact support* in Ω. (This means that g is equal to 0 near the boundary of Ω; more formally, the closure of the set of points at which g is different from 0 is a compact subset of Ω.)

   By changing variables in (1), differentiating under the integral sign, and changing variables back again, show that

   $$\frac{\partial f}{\partial\overline{z}} = -\frac{1}{\pi}\iint_\Omega \frac{\partial g/\partial\overline{\zeta}}{\zeta - z}\, d\xi\, d\eta, \qquad z \in \Omega, \quad \zeta = \xi + i\eta. \tag{2}$$

2. Now apply to the compactly supported function g the inhomogeneous Cauchy integral formula (which you read about in Appendix A) to deduce from equation (2) that $\partial f/\partial\overline{z} = g$ (in the special case that g has compact support in Ω).

The general case follows from the case of compactly supported g by the following device. Fix a point z_0 in Ω, and let φ be a differentiable bump function that is equal to 1 in a neighborhood of z_0 and equal to 0 outside a larger neighborhood of z_0. Rewrite equation (1) as follows:

$$f(z) = -\frac{1}{\pi}\iint_\Omega \frac{g(\zeta)\varphi(\zeta)}{\zeta - z}\, d\xi\, d\eta - \frac{1}{\pi}\iint_\Omega \frac{g(\zeta)(1-\varphi(\zeta))}{\zeta - z}\, d\xi\, d\eta. \tag{3}$$

3. Observe that the $\overline{z}$ derivative of the second integral in (3) equals 0 for z in a neighborhood of z_0, because one can differentiate under the integral sign. (Why does the same reasoning not apply to the first integral?)

4. Use the already proved case of compactly supported functions to see that the first integral in (3) has $\overline{z}$ derivative equal to $g(z)$ for z near z_0. Conclude (since z_0 is arbitrary) that if f is defined by equation (1), then $\partial f/\partial\overline{z} = g$ whether or not g has compact support.