Exercise on Picard's theorems

The goal of this exercise is to prove Picard's little theorem without using the modular function. This proof, different from the one in the textbook, has the advantage that it yields Picard's great theorem too. The proof is not self-contained, however, for it requires the following deep result, a fundamental theorem of Montel: *The family of holomorphic functions on an open set in the plane whose range omits the values* 0 *and* 1 *is a normal family.* (Here normality is understood in the generalized sense that the constant ∞ is an allowed limit function of a subsequence.)

1. The values 0 and 1 are just a convenient normalization: any two distinct complex numbers a and b would serve as well. Why?

Theorem (Picard's little theorem). *The range of a nonconstant entire function cannot omit two values.*

2. To prove Picard's little theorem, suppose that f is an entire function whose range does omit two values, and consider the family of entire functions $\{f_n\}_{n=1}^{\infty}$ defined by $f_n(z) = f(nz)$. Use Montel's theorem to deduce that f must be constant.

Theorem (Picard's great theorem). *In every (punctured) neighborhood of an essential singularity, a holomorphic function assumes every complex value infinitely often, with one possible exception.*

In other words, if f is holomorphic in a punctured disc, and if the isolated singularity at the center of the disc is an essential singularity, then the range of f omits at most one value; and no matter how much one shrinks the punctured disc, the range of the restriction of f to the smaller disc still omits at most one value.

3. To prove Picard's great theorem, assume without loss of generality that the essential singularity is at the origin, and consider the family of functions $\{f_n\}_{n=1}^{\infty}$ defined by $f_n(z) = f(z/n)$. Use Montel's theorem (and remember to consider the possibility of a limit function equal to the constant ∞).

The second theorem subsumes the first one, because a nonconstant entire function has at infinity either an essential singularity or a pole; in the latter case the function is a polynomial.