An entire function of intermediate growth

Answering Jan Cameron's question, Dakota Blair suggested the following example of an entire function that grows faster than any polynomial but that has order zero:

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{n!}\right).$$

Since the function has infinitely many zeroes, it is not a polynomial; consequently, as John Paul Ward pointed out, the function must grow faster than any polynomial, by the version of Liouville's theorem in Exercise 7.8 in the textbook. On the other hand, since $\sum_n (n!)^{-\epsilon}$ converges for every positive ϵ, the function has order zero.[1]

If $M(r)$ denotes the maximum of the modulus of our function on a circle of radius r, then evidently

$$\log M(r)=\sum_{n=1}^{\infty}\log\left(1+\frac{r}{n!}\right).$$

The preceding general considerations show that $\log M(r)$ must grow faster than $k\log r$ for every positive constant k but slower than r^{ϵ} for every positive constant ϵ. In class, I tried unsuccessfully to show that $\log M(r)$ grows like $(\log r)^2$, and it turns out that the true growth rate of $\log M(r)$ is very slightly slower. In the following proposition, the symbol $\sim$ means that the ratio of the two expressions has limit 1.

Proposition. *We have the asymptotic relation*

$$\sum_{n=1}^{\infty}\log\left(1+\frac{r}{n!}\right)\sim\frac{(\log r)^2}{2\log\log r}\qquad \text{as } r\to\infty.$$

Proof. The idea is to break the series at a suitable place and to make different estimates (both from above and from below) on the two sums. To simplify the expression on the right-hand side, it is convenient to replace the variable r

[1] We did not actually prove in class that the order of a canonical infinite product equals the convergence exponent of the zeroes, but this general property of infinite products follows from a small change in the argument that solves problem 4 on the third take-home examination.

by $\exp\sqrt{s}$. Then we have to show that

$$\sum_{n=1}^{\infty} \log\left(1+\frac{e^{\sqrt{s}}}{n!}\right) \sim \frac{s}{\log s} \qquad \text{as } s\to\infty.$$

Let N be the unique integer such that

$$N! \le e^{\sqrt{s}} < (N+1)!$$

(where N depends on s). This integer is a good place to break the series.

Since $\log(1+x) < x$ when $x>0$, we have that

$$0 < \sum_{n=N+1}^{\infty} \log\left(1+\frac{e^{\sqrt{s}}}{n!}\right) < \sum_{n=N+1}^{\infty} \frac{e^{\sqrt{s}}}{n!}.$$

By the choice of N, all the terms of the series on the right-hand side are less than 1, and each term is less than half the preceding one. Consequently, the series is bounded above by $\sum_{j=0}^{\infty} 1/2^j$, which converges to 2, a bound that is independent of N and s. Using Landau's O and o notation (see section 2B of the textbook), we can say that

$$\sum_{n=1}^{\infty} \log\left(1+\frac{e^{\sqrt{s}}}{n!}\right) = O(1) + \sum_{n=1}^{N} \log\left(1+\frac{e^{\sqrt{s}}}{n!}\right).$$

If $n\le N$, then $1+(\exp\sqrt{s}\,)/n! \le 2(\exp\sqrt{s}\,)/n!$, so

$$\sum_{n=1}^{N} \log\left(\frac{e^{\sqrt{s}}}{n!}\right) < \sum_{n=1}^{N} \log\left(1+\frac{e^{\sqrt{s}}}{n!}\right) < N\log 2 + \sum_{n=1}^{N} \log\left(\frac{e^{\sqrt{s}}}{n!}\right).$$

Combining this inequality with the preceding equation shows that

$$\sum_{n=1}^{\infty} \log\left(1+\frac{e^{\sqrt{s}}}{n!}\right) = O(N) + N\sqrt{s} - \sum_{n=1}^{N} \log(n!).$$

Notice that the term $N\sqrt{s}$ cannot be included in the term $O(N)$, because s is not a constant (and N depends on s).

In estimating the remaining sum, I use that $\log(n!) = n\log n + O(n)$, which results from the easy part of Stirling's formula (section 34D in the textbook). Since $\sum_{n=1}^{N} n = O(N^2)$, we then have that

$$\sum_{n=1}^{\infty} \log\left(1+\frac{e^{\sqrt{s}}}{n!}\right) = O(N^2) + N\sqrt{s} - \sum_{n=1}^{N} n\log n.$$

The function $x \log x$ is a monotonically increasing positive function when $x > 1$, so comparing our sum with the area under a graph shows that

$$\int_1^N x \log x \, dx < \sum_{n=1}^N n \log n < \int_2^{N+1} x \log x \, dx.$$

Since $\int x \log x \, dx = \frac{1}{2}x^2 \log x - \frac{1}{4}x^2$, it follows that

$$\sum_{n=1}^N n \log n = \tfrac{1}{2}N^2 \log N + O(N^2),$$

and therefore

$$\sum_{n=1}^\infty \log\left(1 + \frac{e^{\sqrt{s}}}{n!}\right) = O(N^2) + N\sqrt{s} - \tfrac{1}{2}N^2 \log N.$$

Finally, we need to make the relationship between N and s explicit. The definition of N says that

$$\log N! \le \sqrt{s} < \log(N+1)!,$$

so (again by Stirling's formula)

$$\sqrt{s} \sim N \log N, \qquad \text{and} \qquad N\sqrt{s} \sim N^2 \log N.$$

Combining this information with our previous growth estimate shows that

$$\sum_{n=1}^\infty \log\left(1 + \frac{e^{\sqrt{s}}}{n!}\right) = \left(\tfrac{1}{2}N^2 \log N\right)(1 + o(1)).$$

To rewrite the right-hand side in terms of s, observe that

$$\tfrac{1}{2}N^2 \log N = \frac{(N \log N)^2}{2 \log N} \sim \frac{s}{2 \log N}.$$

Moreover, since $\sqrt{s} \sim N \log N$, it follows that

$$\tfrac{1}{2}\log s - \log N - \log\log N \to 0, \qquad \text{so} \qquad \log s \sim 2 \log N.$$

Thus

$$\sum_{n=1}^\infty \log\left(1 + \frac{e^{\sqrt{s}}}{n!}\right) = \frac{s}{\log s}(1 + o(1)),$$

as claimed. □