

**Instructions** Do any five of the following six problems. You may consult the textbook but not other sources; in particular, you may not ask another person for help solving the problems. You may cite and use results proved in the textbook or in class. Please submit your solutions to me in my office (Milner 202) before 4:00 P.M. on Monday, March 26.

1. (a) Prove that if the function  $u$  is harmonic on the open disk<sup>1</sup> of radius  $R$  centered at 0, and if  $u$  takes only *positive* values, then

$$\frac{R-r}{R+r}u(0) \leq u(re^{i\theta}) \leq \frac{R+r}{R-r}u(0), \quad \theta \text{ arbitrary, } 0 < r < R.$$

This useful inequality is known as *Harnack's inequality* for positive<sup>2</sup> harmonic functions.

You may cite (without proof) the result of a homework exercise that you solved previously: namely, the Poisson kernel for the disk of radius  $R$  is equal to

$$\frac{R^2 - r^2}{R^2 + r^2 - 2rR \cos(\theta - \phi)}.$$

- (b) Prove that if  $K$  is a compact subset of a region  $D$ , then there exists a number  $c$  (depending on both  $K$  and  $D$ ) such that for every positive harmonic function  $u$  on  $D$ , one has

$$\max\{u(z) : z \in K\} \leq c \min\{u(z) : z \in K\}.$$

This says that the values of a positive harmonic function are all comparable to each other on a compact set. The point here is that the number  $c$  is independent of the function  $u$ .

Suggestion: cover  $K$  by disks of radius less than half the distance to the boundary of  $D$ .

<sup>1</sup>As mentioned on the first exam, there is little loss of generality in assuming that  $u$  is harmonic on the closed disk. One can work on a disk of radius  $R - \epsilon$  and let  $\epsilon \rightarrow 0$  at the end of the argument.

<sup>2</sup>It would be almost the same to assume that  $u$  is merely non-negative (rather than strictly positive). By the minimum principle for harmonic functions, a non-negative harmonic function cannot take the value 0 in the open disk unless the function is identically equal to 0.

2. This problem generalizes the mean-value property of harmonic functions. Suppose that  $u$  is a harmonic function on the open annulus  $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$ , where  $0 \leq r_1 < r_2 \leq \infty$ . Let  $A(r)$  denote the average of  $u$  around the circle of radius  $r$ , that is,

$$A(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta, \quad r_1 < r < r_2.$$

Prove that there are constants  $a$  and  $b$  such that  $A(r) = a + b \log r$ .

[One approach is to show that  $rA'(r)$  is constant by applying Green's theorem to a region bounded by two concentric circles. The chain rule shows that on a circle with center 0, the differential  $\frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx$  is equal to  $\frac{\partial u}{\partial r} r d\theta$ , which can also be written  $\frac{\partial u}{\partial \mathbf{n}} ds$ .]

3. Prove the following proposition, which is a special case of a result known as Bôcher's theorem.

If  $u$  is a non-negative harmonic function on the punctured closed disk  $\{z \in \mathbb{C} : 0 < |z| \leq 1\}$  that equals 0 on the boundary circle (that is,  $u(z) = 0$  when  $|z| = 1$ ), then there is a non-negative constant  $b$  such that  $u(z) = b \log(1/|z|)$ .

Hints:

- First deduce from the preceding problem that the *average* of  $u$  on a circle of radius  $r$  has the form  $b \log(1/r)$ .
- We may as well assume that  $u$  is strictly positive in the open punctured disk. Why?
- Deduce from the first problem (about Harnack's inequality) that there is some positive number  $c$  such that  $u(z) \geq c \log(1/|z|)$  in the punctured disk. (Take the compact set  $K$  to be the circle of radius  $1/2$  and observe via dilation that the same constant works for any smaller circle.)
- There is a maximal value of  $c$  for which  $u(z) \geq c \log(1/|z|)$  in the punctured disk. Show that for this maximal  $c$ , the inequality must be an equality. (Otherwise the same argument as in the preceding part would produce a bigger  $c$ .)

4. This problem responds to a question raised in class about the nature of isolated singularities of harmonic functions. It turns out that the two examples we considered in class essentially exhaust the possibilities.

More precisely, if  $u$  is a (real-valued) harmonic function in a punctured neighborhood of 0, then one of the following three cases holds.

- (i) The harmonic function  $u$  is bounded in a punctured neighborhood of 0. In this case, the isolated singularity is removable.
- (ii) We have  $\lim_{z \rightarrow 0} |u(z)| = \infty$ . In this case, there is a non-zero constant  $b$  such that  $u(z) - b \log(1/|z|)$  is a harmonic function with a removable singularity at 0. (This is Bôcher's theorem.)
- (iii) If neither of the preceding cases occurs, then there is one sequence of points tending to 0 along which  $|u|$  is unbounded and another sequence along which  $|u|$  is bounded. In this situation,  $u$  assumes every real value infinitely often.

Part (i) is established in section 20D of the textbook. Your job is to handle parts (ii) and (iii).

Hints:

- (a) In part (ii), we may as well work on the unit disk, and we can assume that  $u$  is a positive function. (Why?) Subtract from  $u$  the Poisson integral of its boundary values; why is this difference a non-negative harmonic function? Apply the special case of Bôcher's theorem considered in the previous problem.
- (b) In part (iii), the range of  $u$  is a connected subset of  $\mathbb{R}$ , so the range is an interval (possibly an unbounded interval). If the range of  $u$  is not all of  $\mathbb{R}$ , then  $u$  is bounded from one side, say below. Then  $u(z) + \log(1/|z|)$  is a function of the type considered in part (ii). Deduce a contradiction.

5. In this problem, you will prove an old theorem of the Danish mathematician and Olympic medalist Harald Bohr (brother of Niels Bohr, the physics Nobelist) stating that if  $\left| \sum_{n=0}^{\infty} a_n z^n \right| \leq 1$  when  $|z| \leq 1$ , then  $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$  when  $|z| \leq 1/3$ , and the number  $1/3$  is the best possible.

The intuition behind the theorem is that moving the absolute values inside the sum should increase the sum (by the triangle inequality), but decreasing  $|z|$  should decrease the sum; these two competing influences should have a balance point. The surprising part of the theorem is that there is a uniform balance point that serves for all functions simultaneously; one might expect a priori that it would be necessary to take  $|z|$  closer and closer to 0 as the function varies.

Generalizations and applications of Bohr's theorem have attracted considerable attention during the past decade, and even two Fields medalists have contributed to the subject.<sup>3</sup>

Your proof should have two parts, as follows.

- (a) Show that the radius  $1/3$  does work for all convergent power series that are bounded by 1. Hint: apply the inequality from problem 5 of the first exam.
- (b) Demonstrate that the radius  $1/3$  is sharp by finding an example showing that the conclusion would fail if the number  $1/3$  were replaced by any larger number. Hint: study the Möbius transformation  $\frac{z-a}{1-az}$  when  $a$  is a real number slightly smaller than 1.

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<sup>3</sup>E. Bombieri and J. Bourgain, A remark on Bohr's inequality, *International Mathematics Research Notices* **2004**, no. 80, 4307–4330.

6. In this problem, you will show that the group of analytic automorphisms of the annulus  $\{z \in \mathbb{C} : r < |z| < R\}$  consists of the rotations  $z \mapsto e^{i\theta}z$ , the inversion  $z \mapsto rR/z$ , and compositions of these transformations.

The main technical difficulty in the proof is that the automorphisms are not known a priori to extend continuously to the boundary of the annulus. One can overcome this difficulty by using the strong form of the Schwarz reflection principle in section 20F of the textbook.

Hints:

- (a) If  $f$  is any automorphism of the annulus, then  $f$  maps the circle  $\{z \in \mathbb{C} : |z| = \sqrt{rR}\}$  to some simple closed curve in the (image) annulus. After composing with the inversion, if necessary, we may assume that the “inside” of the circle maps to the “inside” of the image curve, and the outside maps to the outside. (Why?)
- (b) After the preceding normalization, if  $\{z_n\}$  is a sequence of points of the annulus such that  $|z_n| \rightarrow R$ , then  $|f(z_n)| \rightarrow R$ . (Why?)
- (c) Since the circle  $\{z \in \mathbb{C} : |z| = R\}$  can be mapped to the real axis by a Möbius transformation, the strong Schwarz reflection principle implies that  $f$  continues analytically across the outer circle. (To what region does  $f$  extend?) Similarly,  $f$  extends across the inner circle.
- (d) Iterate the reflection process to deduce that  $f$  extends to an automorphism of the punctured plane.
- (e) Argue that the origin is a removable singularity, and  $f$  extends to an automorphism of the whole plane fixing the origin. Hence  $f$  is a rotation.