

Instructions Do any five of the following six problems. You may consult the textbook but not other sources; in particular, you may not ask another person for help solving the problems. You may cite and use results proved in the textbook or in class. Please submit your solutions to me in my office (Milner 202) before 4:00 P.M. on Friday, April 27.

1. In this problem, you will show how to compute the order of an entire function directly from the Maclaurin series coefficients. Recall that if f is an entire function, then $M(r)$ denotes $\max\{|f(z)| : |z| = r\}$, and the order λ of f is defined as follows:

$$\lambda = \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r}.$$

Suppose that $f(z) = \sum_{n=0}^{\infty} c_n z^n$, and define a number β as follows:

$$\beta = \limsup_{n \rightarrow \infty} \frac{n \log n}{\log \frac{1}{|c_n|}} \quad (\text{interpret the fraction as 0 if } c_n = 0).$$

Your task is to prove that in fact $\beta = \lambda$.

Suggestion Prove separately (a) that $\beta \leq \lambda$ and (b) that $\lambda \leq \beta$.

- (a) Fix a positive ϵ . By the definition of order, $M(r) < e^{r^{\lambda+\epsilon}}$ for sufficiently large r . Bound $|c_n|$ for large n by applying Cauchy's estimate with $r = n^{1/(\lambda+\epsilon)}$, and deduce that $\beta \leq \lambda + \epsilon$. Let $\epsilon \downarrow 0$.
- (b) Fix a positive ϵ . Deduce from the definition of β that $|c_n| < n^{-n/(\beta+\epsilon)}$ for sufficiently large n , and use the triangle inequality to bound $M(r)$ by $\sum_{n=0}^{\infty} |c_n| r^n$. By splitting the sum at an integer n close to $(2r)^{\beta+\epsilon}$, show that $\lambda < \beta + 2\epsilon$. Let $\epsilon \downarrow 0$.

Remark The expression for the order in terms of the series coefficients makes it easy to write down an entire function of any prescribed positive order a : namely, the series $\sum_{n=1}^{\infty} z^n / n^{n/a}$ does the job. Moreover, the series $\sum_{n=1}^{\infty} z^n / n^{n^2}$ represents a non-polynomial entire function of order 0.

2. You know that there exists an entire function with prescribed zeroes of prescribed orders, subject to the necessary condition that the zeroes do not accumulate [Weierstrass]. You can immediately deduce by taking the reciprocal that there exists a meromorphic function in the plane with prescribed poles of prescribed orders.

More is true: it is possible to prescribe not only the locations and the orders of the poles of the meromorphic function but also the *principal part* of the Laurent series at each pole. More precisely, suppose that $\{z_j\}_{j=1}^{\infty}$ is a sequence of distinct points having no accumulation point in the finite plane, and $\{p_j\}_{j=1}^{\infty}$ is a sequence of polynomials. You can think of $p_j(\frac{1}{z-z_j})$ as being the principal part of a Laurent series based at the point z_j . *Mittag-Leffler's theorem* says that there exists a function f meromorphic in the plane, with poles only at the points z_j , such that for each j , the function $f(z) - p_j(\frac{1}{z-z_j})$ has a removable singularity at z_j . In other words, the function f has the specified principal parts at the specified poles.

Your task in this problem is to prove Mittag-Leffler's theorem.

Hint The first try is $\sum_{j=1}^{\infty} p_j(\frac{1}{z-z_j})$, but this series need not converge. Show, however, that there exist polynomials q_j such that the series $\sum_{j=1}^{\infty} \left(p_j(\frac{1}{z-z_j}) - q_j(z) \right)$ does converge (uniformly on compact sets that avoid the sequence of poles). In fact, $q_j(z)$ can be taken to be a suitable partial sum of the Maclaurin series of $p_j(\frac{1}{z-z_j})$.

Remarks Notice that the theorem of Mittag-Leffler uses infinite series and additive convergence factors, while the theorem of Weierstrass uses infinite products and multiplicative convergence factors.

Mittag-Leffler proved much more than is stated above.¹ First of all, an analogous theorem holds on a general planar region. Secondly, one can simultaneously prescribe the locations of both the zeroes and the poles of a meromorphic function. Thirdly, one can prescribe any finite number of terms of the Laurent series (not just the terms containing negative powers).

¹G. Mittag-Leffler, “Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante,” *Acta Math.* 4 (1884), 1–79.

3. You know Montel's theorem: a family of analytic functions on a planar region is a normal family (in the sense that every sequence of functions in the family admits a subsequence that converges uniformly on compact sets to an analytic function) if and only if the family is locally bounded, that is, for every compact subset K there is a constant C such that $|f(z)| \leq C$ for all z in K and every f in the family.

In this problem, you are to prove *Marty's theorem*, which gives a necessary and sufficient condition for a family of analytic functions on a planar region to be a normal family in the extended sense that every sequence of functions in the family admits either a subsequence that converges uniformly on compact sets to an analytic function or a subsequence that converges uniformly on compact sets to the constant function ∞ . Marty's criterion is that for every compact subset K there is a constant C such that $\frac{|f'(z)|}{1 + |f(z)|^2} \leq C$ for all z in K and every f in the family.

Hints If the family is normal, but Marty's criterion is violated, then there exists a compact set and a sequence of functions such that ...; pass to a convergent subsequence and deduce a contradiction.

In the converse direction, the key issue in deducing normality from Marty's criterion is to prove some uniform control: you need to know that if $|f(z)|$ is small (respectively large) at a point, then $|f(z)|$ remains small (respectively large) in a neighborhood of the point (the neighborhood being independent of the function). One way to get this control is to compute $\frac{\partial}{\partial r} \tan^{-1} |f(z_0 + re^{i\theta})|$ by the chain rule and deduce that

$$|\tan^{-1} |f(z_0 + Re^{i\theta})| - \tan^{-1} |f(z_0)|| \leq \int_0^R \frac{|f'(z_0 + re^{i\theta})|}{1 + |f(z_0 + re^{i\theta})|^2} dr.$$

Remark The quantity $\frac{|f'(z)|}{1 + |f(z)|^2}$, sometimes called the spherical derivative, has the useful property that it is unchanged when f is replaced by its reciprocal $1/f$. Marty's criterion is local boundedness of the family of spherical derivatives.

4. The word “order” appears in several different mathematical contexts, one of them being the theory of entire functions, and the same remark applies to the word “genus”. If f is an entire function, and

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2} \left(\frac{z}{z_n}\right)^2 + \cdots + \frac{1}{p} \left(\frac{z}{z_n}\right)^p\right),$$

where g is a polynomial and p is the minimal non-negative integer for which the infinite product converges, then the *genus* of f is the maximum of p and the degree of g . (Set p equal to 0 if the infinite product already converges without the exponential factors; in particular, $p = 0$ if the function has only a finite number of zeroes.)

Hadamard’s factorization theorem, which we proved in class, can be rephrased as the statement that *the genus of an entire function does not exceed its order*.

Your task in this problem is to prove a sort of converse to Hadamard’s theorem: namely, *the order of an entire function does not exceed the genus by more than 1*.

Hints The main difficulty is estimating the rate of growth of the infinite product. Apply Exercise 31.2 in the textbook to show that the typical term in the product is dominated by $\exp|z/z_n|^{p+1}$ when $|z/z_n| < 1/2$. Use elementary estimates to show that there is a constant c such that the typical term in the product is dominated by $\exp c|z/z_n|^p \leq \exp 2c|z/z_n|^{p+1}$ when $|z/z_n| \geq 1/2$. Then recall from class that p is the smallest non-negative integer for which the series $\sum_n 1/|z_n|^{p+1}$ converges.

Remark Combining the italicized statements above shows that

$$\text{genus} \leq \text{order} \leq 1 + \text{genus}.$$

This double inequality is sharp: examples show that the order can assume any of the allowed values.

5. In this problem, you are to supply proofs (modulo one ingredient) of Picard's theorems about the range of analytic functions. The ingredient that you may assume without proof is the following result of Montel.

Montel's fundamental normality criterion *Suppose a and b are two distinct complex numbers. The family of analytic functions on a region that take neither the value a nor the value b is a normal family in the region.* Here normality is understood in the extended sense that the constant function ∞ is admitted as a possible limit function.

(i) **Picard's little theorem** *The range of a nonconstant entire function cannot omit two values.*

For the proof, suppose that f is an entire function whose range does omit two values, and consider the family of entire functions $\{f_n\}_{n=1}^{\infty}$ defined by $f_n(z) = f(nz)$. Apply Montel's fundamental normality criterion to deduce that f must be constant.

(ii) **Picard's great theorem** *In every punctured neighborhood of an essential singularity, an analytic function assumes every complex value—with one possible exception—infinitely often.*

For the proof, assume without loss of generality that the essential singularity is at the origin, and suppose there are two values each taken only a finite number of times. After shrinking the neighborhood, one can assume that there are two omitted values. Consider the family of functions $\{f_n\}_{n=1}^{\infty}$ defined by $f_n(z) = f(z/n)$, and apply Montel's criterion, remembering to account for the possibility of a limit function equal to the constant ∞ .

Remark Picard's great theorem subsumes the little theorem, for an entire function either has an essential singularity at infinity or is a polynomial (and a nonconstant polynomial assumes every complex value). Thus one can strengthen the little theorem to say that a non-polynomial entire function takes every complex value infinitely often, with one possible exception.

6. Suppose D is a bounded, simply connected planar region, and let p be a point in D . Consider the family of analytic functions f on D such that f maps D into (not necessarily onto) the unit disk, $f(p) = 0 = f'(p)$, and $f''(p)$ is a positive real number.
- Show that there is an extremal function that maximizes the value of $f''(p)$ within the indicated class.
 - Show that the extremal function is unique by showing that it is equal to the square of the Riemann mapping function.