In this assignment, you will show that every biholomorphic map (that is, bijective analytic function) between two annuli is necessarily a linear fractional transformation (Möbius transformation): namely, a composition of translations, dilations, rotations, and inversions. Moreover, you will show that two annuli are conformally equivalent if and only if they have the same ratio of outer radius to inner radius.

The main technical difficulty is that a biholomorphic map between annuli is not known a priori to extend continuously to the boundary. You can overcome this difficulty by using the strong form of the Schwarz reflection principle that does not require continuity at the boundary.

1. Show that for every annulus, there is some radius R such that the annulus can be mapped via a Möbius transformation to the annulus $\{\, z \in \mathbb{C} : 1 < |z| < R \,\}$. Let A_R denote this model annulus.

2. Observe that a rotation is a conformal automorphism of A_R, as is the inversion $z \mapsto R/z$. (It will follow at the conclusion of this assignment that rotations and the inversion generate all the automorphisms of A_R.)

3. Suppose that $f : A_R \to A_S$ is a biholomorphic mapping. (It is going to turn out that the existence of such a mapping forces R to equal S.) Suppose C is a circle in A_R with center at the origin. Argue that f maps the part of A_R lying inside C either to the part of A_S lying inside the image curve $f(C)$ or to the part of A_S lying outside $f(C)$. Reduce to the former case by composing f with an inversion, if necessary.

4. Deduce that after the preceding normalization, if $\{z_n\}$ is a sequence of points in A_R such that $|z_n| \to 1$, then $|f(z_n)| \to 1$. Similarly, if $|z_n| \to R$, then $|f(z_n)| \to S$.

5. Since every circle can be mapped to the real axis by a Möbius transformation, the Schwarz reflection principle implies that f continues analytically across the outer boundary of A_R. To what region does f extend? Similarly, f extends across the inner circle.

6. Iterate the reflection process to deduce that f extends to be an automorphism of the punctured plane.

7. Argue that the origin is a removable singularity, and f extends to an automorphism of the whole plane fixing the origin. Hence the normalized function f is a linear function.

8. Deduce that $R = S$, and the original function f is a linear fractional map.