In this assignment, you will prove Picard's theorems by applying Montel's fundamental normality criterion. (The criterion states that a family of holomorphic functions from a connected open set into $\mathbb{C} \setminus \{0, 1\}$ is normal in the extended sense that the constant function ∞ is an allowed limit.)

Picard's "little" theorem describes the range of an entire function. The case of a polynomial of positive degree n is covered by the fundamental theorem of algebra: the polynomial assumes every complex value exactly n times, counting multiplicity. An entire function that is not a polynomial is called *transcendental*.

Theorem (Picard's little theorem)**.** *A transcendental entire function assumes every complex value—with one possible exception—infinitely often.*

1. The exceptional value—if there is one—might be completely omitted or might be taken a finite number of times.
   a) Give an example of a transcendental entire function that omits the value 0.
   b) Give an example of a transcendental entire function that takes the value 0 exactly five times (counting multiplicity).
   c) Give an example of a transcendental entire function that takes every value infinitely many times.

Theorem (Picard's great theorem)**.** *A holomorphic function takes every value—with one possible exception—infinitely often in every (punctured) neighborhood of an essential singularity.*

Thus if f is holomorphic in a punctured disk, and if the range of f omits two values, then the singularity cannot be an essential singularity: the singularity can only be a pole or a removable singularity.

2. To prove Picard's great theorem, consider the contrapositive statement. Suppose that f is holomorphic in some punctured neighborhood of the origin, and there are two distinct values that f assumes only a finite number of times. The goal is to show that f does not have an essential singularity.
   a) Why is there no loss of generality in supposing that the two exceptional values are 0 and 1?
   b) Observe that there is some positive radius δ such that f is holomorphic in the punctured disk $\{ z \in \mathbb{C} : 0 < |z| < \delta \}$ and does not assume either of the exceptional values in this region.
   c) For each positive integer n, set $f_n(z)$ equal to $f(z/n)$. Apply Montel's fundamental normality criterion to the family $\{ f_n \}$ on the punctured disk $\{ z \in \mathbb{C} : 0 < |z| < \delta \}$ to deduce that the singularity at the origin is either a removable singularity or a pole.

3. Deduce Picard's little theorem as a corollary of Picard's great theorem, as follows. If $f(z)$ is an entire function, then $f(1/z)$ has an isolated singularity at the origin.
   a) Show that if $f(1/z)$ has a removable singularity at the origin, then the entire function $f(z)$ must be constant.
   b) Show that if $f(1/z)$ has a pole at the origin, then the entire function $f(z)$ must be a polynomial.
   Suggestion: A generalization of Liouville's theorem states that if an entire function grows slower than some polynomial, then the function *is* a polynomial. You can prove this generalization by applying Cauchy's estimates for derivatives.
   c) Conclude that Picard's little theorem follows from Picard's great theorem.
   (The catchphrase is that a transcendental entire function "has an essential singularity at infinity.")