In this assignment, you will apply subharmonic functions to prove Hadamard's three-circles theorem (which is one of the topics on the official syllabus for the qualifying examination in complex analysis).

1. Suppose u is a subharmonic function that is bounded above in a vertical strip, say $\{ x+iy : a < x < b,\ y \in \mathbb{R} \}$. Let $m(x)$ denote $\sup\{ u(x+iy) : y \in \mathbb{R} \}$ (the supremum of the values of u on a vertical line). Show that $m(x)$ is a convex function on the interval (a,b). In other words, if x_1 and x_2 are two points in the interval (a,b), and x is a point in between, then

   $$m(x) \le \lambda m(x_1) + (1-\lambda)m(x_2), \qquad \text{where } x = \lambda x_1 + (1-\lambda)x_2, \text{ and } 0 < \lambda < 1.$$

   An equivalent statement is that

   $$m(x) \le \frac{x_2 - x}{x_2 - x_1} m(x_1) + \frac{x - x_1}{x_2 - x_1} m(x_2) \qquad \text{when} \quad x_1 < x < x_2.$$

   Suggestion: A first-degree polynomial in x is, in particular, a harmonic function that you can compare to the subharmonic function $u(x+iy)$. You need a bounded region on which to apply the maximum principle, so suppose initially that $u(x+iy) \to -\infty$ uniformly with respect to x when $|y| \to \infty$, and apply the maximum principle on a suitable rectangle. To handle the general case, consider functions like $u(z) - \varepsilon \operatorname{Re} \cos(\frac{z-a}{b-a})$.

2. Deduce that if f is a bounded holomorphic function in a vertical strip, and $M(x)$ denotes $\sup\{ |f(x+iy)| : y \in \mathbb{R} \}$, then

   $$M(\lambda x_1 + (1-\lambda)x_2) \le M(x_1)^{\lambda} M(x_2)^{1-\lambda} \qquad \text{when} \qquad 0 < \lambda < 1.$$

   This property of the function M is called *logarithmic convexity*, and the result is called *Hadamard's three-lines theorem*.

3. Suppose u is a subharmonic function in an annulus, say $\{ z \in \mathbb{C} : a < |z| < b \}$. Let $m(r)$ denote $\sup\{ u(z) : |z| = r \}$. Show that if $a < r_1 < r < r_2 < b$, then

   $$m(r) \le \frac{\log r_2 - \log r}{\log r_2 - \log r_1} m(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} m(r_2).$$

   One sometimes sees the statement that $m(r)$ is "a convex function of $\log r$." An equivalent formulation of the inequality is that

   $$m(r) \le \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}} m(r_1) + \frac{\log \frac{r}{r_1}}{\log \frac{r_2}{r_1}} m(r_2).$$

   Suggestion: Either use a harmonic comparison function of the form $A + B \log|z|$, or apply part 1 to the composite function $u(e^z)$.

4. Deduce that if f is a holomorphic function in the annulus $\{ z \in \mathbb{C} : a < |z| < b \}$, and $M(r)$ denotes $\sup\{ |f(z)| : |z| = r \}$, then

$$M(r) \le M(r_1)^{\lambda} M(r_2)^{1-\lambda}, \qquad \text{where } a < r_1 < r < r_2 < b, \text{ and } \lambda = \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}}.$$

This statement is *Hadamard's three-circles theorem*.