You know that every biholomorphic automorphism of the open unit disk can be written in the following form:

$$z \mapsto e^{i\theta}\frac{a-z}{1-\overline{a}z}, \qquad \text{where } \theta \in \mathbb{R} \text{ and } |a| < 1.$$

(One proof is to compose an arbitrary automorphism with one of these special ones to fix the origin; then apply the Schwarz lemma. Another proof is to use the Schwarz reflection principle to show that every automorphism of the disk extends by symmetry to the exterior of the disk and consequently has to be a Möbius transformation.)

By Montel's theorem, the family of automorphisms is a relatively compact subset of the metric space of all holomorphic functions on the unit disk (since the automorphisms are a bounded family). In this assignment, you will determine the limit points of the set of automorphisms.

1. Suppose that $\{f_n\}$ is a sequence of automorphisms that converges uniformly on compact subsets of the open unit disk to a holomorphic function f. Show that f is either an automorphism or a constant function.
   Suggestion: Apply the circle of ideas around Hurwitz's theorem. Notice that the sequence $\{f_n^{-1}\}$ of inverses is a normal family too.

2. Suppose that $\{\theta_n\}$ is a sequence of real numbers such that $\theta_n \to \theta$, and $\{a_n\}$ is a sequence of points of the open unit disk such that $a_n \to a$. (Here the limit point a might be either in the interior or on the boundary of the disk.)

   a) Show that the sequence of automorphisms $\left\{e^{i\theta_n}\frac{a_n - z}{1-\overline{a_n}\,z}\right\}$ converges, uniformly on compact subsets of the open unit disk, to the function $e^{i\theta}\dfrac{a-z}{1-\overline{a}z}$.

   b) Observe that when $|a| = 1$, this limit function is constant on the open unit disk. What is the constant value?

   c) In part 2a, is the convergence uniform on the *closed* unit disk?

3. Passing to a suitable subsequence in part 1 reduces to the situation in part 2. Deduce that the limit function f in part 1, if not an automorphism, is a constant of modulus equal to 1.