In this assignment, you will deepen your understanding of the concept of the order of an entire function by comparing three definitions.

Recall that $M_f(r)$, or $M(r)$ for short, denotes $\max\{\,|f(z)| : |z| \le r\,\}$. If f is a nonconstant entire function, then Liouville's theorem implies that $M(r) \to \infty$ as $r \to \infty$. The order of f characterizes how fast $M(r)$ goes to ∞.

Consider the following three numbers associated to a nonconstant entire function f whose Maclaurin series is $\sum_{n=0}^{\infty} c_n z^n$:

$$\rho := \inf\left\{ t \in \mathbb{R} : \lim_{r\to\infty} \frac{\log M(r)}{r^t} = 0 \right\}$$
$$\lambda := \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}$$
$$\beta := \limsup_{n\to\infty} \frac{n \log n}{\log \frac{1}{|c_n|}}.$$

(In the definition of β, interpret the fraction as 0 when $c_n = 0$.) Your main task is to show that $\rho = \lambda = \beta$.

1. To check that you understand the definitions, verify by explicit calculation that when $f(z) = ze^z$, the three numbers ρ, λ, and β all are equal to 1.

2. To see that there is some point in having different formulations of the concept of order, let s be an arbitrary positive real number, and suppose that

   $$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/s}}.$$

   Use the third definition to show that f is an entire function of order s. (Thus there exists an entire function of an arbitrary, prescribed positive order.)

Now let f be a general nonconstant entire function, and fix a positive ε.

3. Observe that $\log M(r) < r^{\rho+\varepsilon}$ for sufficiently large r. Deduce that $\lambda \le \rho + \varepsilon$.

4. Observe that $\log M(r) < r^{\lambda+\varepsilon}$ for sufficiently large r. Deduce that $\rho \le \lambda + 2\varepsilon$.

5. Observe that $M(r) < e^{r^{\lambda+\varepsilon}}$ for sufficiently large r. Bound $|c_n|$ for large n by applying Cauchy's estimate with $r = n^{1/(\lambda+\varepsilon)}$, and deduce that $\beta \le \lambda + \varepsilon$.

6. Show that $|c_n| < n^{-n/(\beta+\varepsilon)}$ for sufficiently large n. Observe that $M(r) \le \sum_{n=0}^{\infty} |c_n|\, r^n$. By splitting the sum where $n \approx (2r)^{\beta+\varepsilon}$, show that $\lambda \le \beta + 2\varepsilon$.

Finally let $\varepsilon \downarrow 0$ to deduce that $\rho = \lambda = \beta$.