1. Prove that if a is a complex number of modulus less than 1, then the function that sends z to $\dfrac{a-z}{1-\bar{a}z}$ is a self-inverse holomorphic bijection of the open unit disk, $\{ z \in \mathbb{C} : |z| < 1 \}$.

   (See Exercises 1.22 and 25.5 in the textbook; also somewhat relevant is the Schwarz lemma on page 119.)

   [Holomorphic functions that have holomorphic inverses are *biholomorphic*. A biholomorphic map from a domain onto itself is a holomorphic *automorphism* of the domain. It follows from the Schwarz lemma that the automorphisms of the unit disk are the maps indicated above and their compositions with rotations.]

2. Prove that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of nonzero points of the open unit disk (repetitions are allowed), then the infinite product

$$\prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \cdot \frac{a_n - z}{1 - \overline{a_n}\, z}$$

   converges absolutely and uniformly on compact subsets of the open unit disk if and only if the infinite series $\sum_{n=1}^{\infty}(1-|a_n|)$ converges.

   [Such a product is called a *Blaschke product* after Wilhelm Blaschke (1885–1962), who worked mainly in geometry. Blaschke directed the dissertation of the great differential geometer Shiing-Shen Chern (1911–2004).]

3. Deduce that there exists a *bounded* holomorphic function on the open unit disk that is singular at every point of the boundary. (A holomorphic function is singular at a boundary point p if there exists no disk centered at p to which the function extends holomorphically.)