Exercise on convergence and disk automorphisms

Recall, or look up in Section 8.4 of the textbook, that if a is a complex number of modulus less than 1, and θ is a real number, then the function that sends z to $e^{i\theta}\dfrac{a-z}{1-\overline{a}z}$ is a holomorphic bijection of the open unit disk, $\{ z \in \mathbb{C} : |z| < 1 \}$. Moreover, every "automorphism" of the unit disk can be expressed this way.

1. Prove that if $(a_n)_{n=1}^{\infty}$ is a sequence of nonzero points of the open unit disk (repetitions are allowed), then convergence of the infinite series $\sum_{n=1}^{\infty}(1-|a_n|)$ is a necessary and sufficient condition for the infinite product

$$\prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \cdot \frac{a_n - z}{1 - \overline{a_n}\, z}$$

to converge absolutely and uniformly on compact subsets of the open unit disk.

[Such a product is called a *Blaschke product* after Wilhelm Blaschke (1885–1962), who worked mainly in geometry. Blaschke directed the dissertation of the great differential geometer Shiing-Shen Chern (1911–2004).]

The family of all automorphisms of the unit disk is not only locally bounded but even bounded, so Montel's theorem implies that the automorphisms form a relatively compact subset of the metric space of all holomorphic functions on the unit disk. The following exercise makes precise how this relatively compact subset fails to be closed.

2. Prove that if a sequence of disk automorphisms converges uniformly on compact subsets of the disk, then the limit function is either an automorphism or a constant function, with the constant being of modulus 1.

Suggestion: Apply the circle of ideas around Hurwitz's theorem.