Exercise on Alice Roth's Swiss cheese

If K is a compact set in $\mathbb{C}$, and f is continuous on K and holomorphic on the interior of K, can f be approximated uniformly on K by rational functions?

S. N. Mergelyan (С. Н. Мергелян) proved that the answer is "yes" when the diameters of the components of the complement of K are bounded away from 0 (in particular, when $\mathbb{C} \setminus K$ has a finite number of components). Runge's theorem implies that the answer is "yes" when f can be approximated uniformly on K by functions holomorphic in an open neighborhood of K. In general, however, the answer is "no".

The Swiss mathematician Alice Roth constructed a counterexample by punching an infinite number of holes in a disk.[1] Such a set is now called a "Swiss cheese" (which is the generic name in English for a pale-yellow cheese having many holes). Your task is to recreate a version of Alice Roth's example, as follows.

1. Demonstrate the existence of a sequence $\{D_j\}_{j=1}^{\infty}$ of open disks satisfying all three of the following properties simultaneously.

   (a) The closures of the disks D_j are pairwise disjoint and are contained in the open unit disk D.

   (b) The sum of the radii of the disks D_j is less than $1/2$.

   (c) If K denotes $\overline{D} \setminus \bigcup_{j=1}^{\infty} D_j$, then K is compact and has empty interior.

2. Use the homology version of Cauchy's theorem to show that if R is a rational function whose poles lie in the complement of K, then

$$\oint_{\partial D} R(z)\,dz = \sum_{j=1}^{\infty} \oint_{\partial D_j} R(z)\,dz.$$

3. Show that $\oint_{\partial D} \overline{z}\,dz = 2\pi i$, and $\left| \sum_{j=1}^{\infty} \oint_{\partial D_j} \overline{z}\,dz \right| < \pi$.

4. Deduce that if $f(z) = \overline{z}$, then f is an example of a continuous function on K that is holomorphic on the interior of K (vacuously, since K has empty interior) and that cannot be arbitrarily well approximated uniformly on K by rational functions.

[1] Alice Roth, Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen, *Commentarii Mathematici Helvetici* **11** (1938) 77–125.