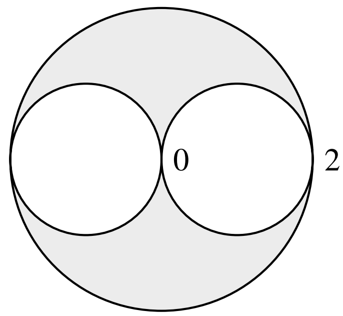
Exercise on Runge's theorem

Carl Runge
(1856–1927)

This exercise is intended to reinforce your understanding of Runge's theorem: *If K is a compact subset of $\mathbb{C}$, then every function that is holomorphic in a neighborhood of K can be approximated uniformly on K by rational functions.* Moreover, if E is a set containing at least one point from each bounded component of $\mathbb{C} \setminus K$, then the poles can be taken to lie in E.

Consider the compact set K shown in the following figure: a closed disk of radius 2 from which two open disks of radius 1 have been removed.



$$K = \{\, z \in \mathbb{C} : |z| \le 2 \text{ and } |z-1| \ge 1 \text{ and } |z+1| \ge 1 \,\}$$

1. Can every function holomorphic in a neighborhood of K be approximated uniformly on K by rational functions with poles only at the points -1 and 1? How about by polynomials?

2. Give an example of a function holomorphic in a neighborhood of K that is not rational, cannot be approximated uniformly on K by polynomials, but can be approximated uniformly on K by rational functions with pole only at the point -1.

3. The *polynomially convex hull* $\widehat{K}$ of a compact set K is the set of points that cannot be "separated" from K by a polynomial: namely, if $\|p\|_K$ denotes $\sup\{\, |p(z)| : z \in K \,\}$, then

$$\widehat{K} = \{\, w \in \mathbb{C} : |p(w)| \le \|p\|_K \text{ for every polynomial } p \,\}.$$

   Determine the polynomially convex hull of the compact set K shown in the figure. Can you generalize to handle an arbitrary compact set?

4. Let G be the interior of the compact set K shown in the figure; that is, $G = K \setminus \partial K$. The open set G has how many components? The complement $\mathbb{C} \setminus G$ has how many components?

5. Can every holomorphic function on G be approximated by polynomials, uniformly on every compact subset of G?

Of course, your answers should be more than "yes" or "no": an explanatory reason is required.