Exercise on subharmonic functions and Perron's method for solving the Dirichlet problem

Introduction

Subharmonicity is a local property that can be characterized as follows. (The three properties stated below are mutually equivalent, modulo assumptions about smoothness of the function.) Suppose G is an open subset of $\mathbb{C}$, and $u\colon\ G \to [-\infty, \infty)$ is an upper semicontinuous function.

Maximum principle The function u is subharmonic on G if and only if for every point z_0 in G and every harmonic function v in a neighborhood of z_0, the difference $u - v$ does not have a (weak) local maximum at z_0, unless $u - v$ is constant on a (possibly smaller) neighborhood of z_0.

An equivalent global statement is that for every connected open subset G_1 of G (possibly G itself) and every harmonic function v on G_1, the difference $u - v$ has no (weak) global maximum at a point of G_1, unless $u - v$ is constant on G_1.

Notice that one of the allowed choices for the comparison function v is the 0 function, so a subharmonic function u itself satisfies a maximum principle.

Sub-mean-value property The function u is subharmonic on G if and only if for every closed disk $\overline{B}(z_0; r)$ in G, the value $u(z_0)$ at the center does not exceed the average value on the boundary: namely,

$$u(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta.$$

(If u is continuous, then the Riemann integral can be used. If u is merely upper semicontinuous, then the Lebesgue integral is needed.)

An equivalent property is that for every point z_0 in G, there is a positive radius r_0 such that the sub-averaging inequality holds on each disk $B(z_0; r)$ for which $0 < r < r_0$.

Positive Laplacian When $u(x, y)$ is twice continuously differentiable, the function $u(x, y)$ is subharmonic if and only if $\Delta u \ge 0$, where Δu is an abbreviation for $u_{xx} + u_{yy}$.

(Even when u is not differentiable in the ordinary sense, subharmonicity is characterized by a weakly positive Laplacian if the derivatives are interpreted in the sense of distributions, or generalized functions.)

1. Examples

One simple example of a subharmonic function is $|z|^2$, or $x^2 + y^2$. Notice that this function is bounded below (by 0) and is convex.

Your first task is to give an example of a subharmonic function on the whole plane $\mathbb{C}$ that is a polynomial in the real coordinates x and y, harmonic on no open subset, and not bounded below.

Second, give an example of a subharmonic function that is continuous on the whole plane $\mathbb{C}$, radial (that is, a function only of $|z|$), non-negative, and not convex.

2. Liouville's theorem revisited

Prove that if a function is subharmonic on the entire plane $\mathbb{C}$ and is bounded above, then the function must be constant.

Suggestion: The function $A + B \log |z|$ is an available harmonic comparison function on every annulus centered at the origin for every choice of the constants A and B.

Remark The statement does indeed generalize Liouville's theorem, for if f is a holomorphic function, then $|f|$ is subharmonic. (Notice that if $|f|$ is constant, then f is too, by the open-mapping theorem for holomorphic functions.) The statement is specific to the plane, for when $n \geq 3$, the space $\mathbb{R}^n$ supports bounded nonconstant subharmonic functions.

3. Failure of the Perron method on the punctured disk

The Perron family for the punctured disk $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$ with given boundary values is the family of all subharmonic functions on the punctured disk whose lim sup at the boundary does not exceed the prescribed boundary values. The lack of a subharmonic peaking function at the origin means that one loses control of the upper envelope of the Perron family.

If the boundary values are equal to 1 when $|z| = 1$ and equal to 0 when $z = 0$, then what is the pointwise supremum of the corresponding Perron family on the punctured disk? Why?

If the boundary values are equal to 0 when $|z| = 1$ and equal to 1 when $z = 0$, then what is the pointwise supremum of the corresponding Perron family on the punctured disk? Why?